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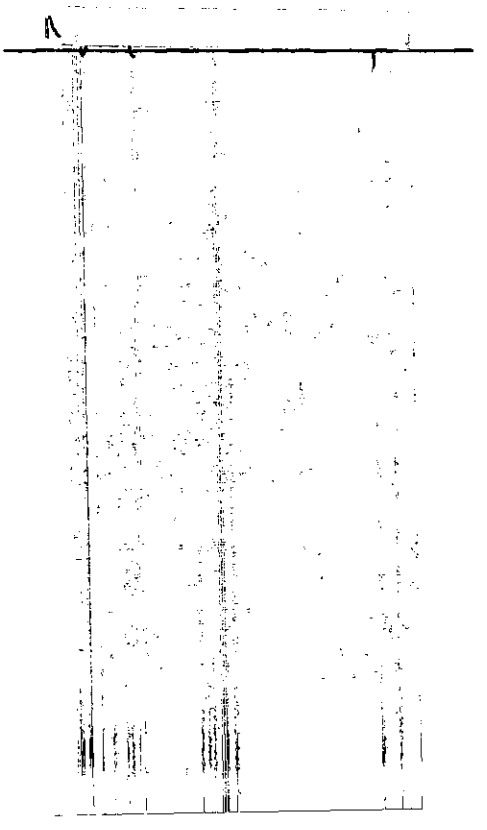

NUCLEAR MATRIX ELEMENTS IN BETA DECAY FOR SPHEROIDAL-SHAPED NUCLEI

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NUCLEAR MATRIX ELEMENTS IN

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On many occasions in his life, the author has realized that intelligence and knowledge play but a minor rôle. In such a situation, all he could do was to look upon his lucky star and his surroundings. To the latter the author feels bound to express his gratitude, particularly:

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

Since the explicit form of nuclear forces is unknown, nuclear properties are often studied in an approximate manner via nuclear models. These nuclear models are then used to calculate the nuclear matrix elements which play an important role in all theoretical expressions for experimental observables such as the transition probability, the spectrum shape correction factor, the beta-gamma angular correlation, etc. These predictions are then compared with experiments and the feasibility of the nuclear model may be determined.

In this investigation, by making use of the unified model, developed mainly by A. Bohr, B. R. Mottelson and S. V. Nilsson, the general expressions for nuclear matrix elements in beta decay are given, together with their selection rules. All interaction operators are written in the form of irreducible spherical tensors, following the notations used by Rose and Osborn. The nucleus is assumed to be strongly deformed, spheroidal-shaped, and constant in shape for a sufficiently long time so that the motion of the nucleons is not appreciably affected. Due to these assumptions, it is shown that an odd-(even-) number particle intrinsic wave function must be used to represent the nuclear state of odd-A (even-A) nuclei.

In the present calculation, one- and two-particle intrinsic wave functions are used for odd-and even-mass nuclei, respectively. It is found that for even-mass nuclei, the transition matrix element depends also on the angular momentum coupling of the last two nucleons,

and this leads to different selection rules.

In particular, the nuclear matrix elements for first-forbidden beta decay in the isotope of Tm^{170} and Re^{186} are evaluated for different values of the deformation parameter η, and of the relative strength μ of the spin-orbit and of the interpolation terms (ℓ^2) in the Nilsson model. It is found that the nuclear parameters for Tm^{170} do not change smoothly as η and μ vary, and do not seem to agree with experiments, while the results for Re^{186} are very consistent and in good agreement with experimental results.

CHAPTER I

INTRODUCTION

In the opening years of 20th century, a host of important facts presented a new challenge in physics. Physicists were forced either to reject classical theory which had been so successful in explaining macroscopic systems, or to provide a new interpretation and new techniques. Among other things, beta radioactivity was discovered. By 1930, the explanation for a physical system of an atomic scale was fairly well-established with the help of a new powerful technique: the quantum theory. The problem of beta decay, however, remained theoretically unanswered. The understanding of the beta decay process has grown rather slowly since it involves a knowledge both of nuclear structure and of the force which produces beta decay. Aside from the theoretical aspect, it also involves experimental techniques which were not available until after the second world war.

With the development of quantum field theory, Fermi[1] in 1934 formulated the fundamental theory of beta decay which, in general, is still used today. While the force producing beta decay is very weak, the nuclear force is so strong that it makes the perturbation calculation impossible. Yet, for a full quantitative description of beta decay, one needs to know the nuclear states, i.e., to obtain the solutions of the Schrodinger equation for the nucleus. The problem of the strong nuclear force is still subject to much elaborate study. In the meantime, in

order to explain and classify the growing accumulation of nuclear data, one chooses to study the nuclear problem in an approximate manner through various nuclear models.

In the next two sections of this chapter a brief exposé of the theory of beta decay and of some nuclear models will be given. The final section will define the scope of this investigation.

Origin and Definition of Beta Decay Nuclear Matrix Elements

The typical beta decay process is

$$n \longrightarrow p + e^- + \bar{\nu} \ . \tag{1-1}$$

Mathematically speaking, the above reaction is equivalent to

$$n + \nu \longrightarrow p + e^- \ . \tag{1-2}$$

Thus, one electron and one proton are created, while one neutrino and one neutron are destroyed. The interacting Hamiltonian can be constructed in analogy with the Hamiltonian for the electromagnetic radiation by writing it as a bilinear form of the states representing the four particles in question, together with certain operators suitably chosen. This Hamiltonian is expected to be independent of the choice of the coordinate systems, i.e., it must be invariant under a Lorentz transformation. As is well known, for spin 1/2 particles, there are five groups of operators , O_x , which transform covariantly under a Lorentz transformation. According to their transformation properties, they are called scalar (S), vector(V), tensor(T), axial-vector (A) and pseudoscalor (P), and usually written in terms of Dirac matrices γ_μ (μ=1,2,3,4,).[2]

Among these five groups of operators, the vector form was originally used by Fermi.

Prior to 1956 it was generally believed that parity was conserved. Therefore, the interacting Hamiltonian was required not only to be invariant under a translation or a rotation but also under a reflection of coordinates. In 1956 the parity non-conservation in weak interactions was suggested by T. D. Lee and C. N. Yang[3], and experimentally confirmed by C. S. Wu et al.[4]. The most general form of the beta interaction consequently may be written

$$H = \sum_x \langle p|O_x|n\rangle\langle e|O_x(C_x + C_x'\gamma_5)|\nu\rangle \delta(\vec{R}-\vec{r}) + h.c. , \qquad (1\text{-}3)$$

where x takes on values S, V, T, A, and P, $\vec{R}$ and $\vec{r}$ represent the nucleon and lepton coordinates respectively, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The δ-function implies a local interaction as in electromagnetic theory. The term "h.c." means Hermitian conjugate and is included to make the Hamiltonian Hermitian. The C's are the coupling constants and may be complex. There are, in total, 20 constants C to be determined.

Experimental evidence shows that the neutrino possesses definite helicity[5] which theoretically implies $C_x = C_x'$. Furthermore, by considering the helicity of the emitted electron[6] when its velocity approaches the speed of light, it is shown that $C_S = C_T = C_P = 0$. Finally, experiments[7] indicate that the ratio of C_A to C_V is real with

$$C_A \sim -1.21\, C_V . \qquad (1\text{-}4)$$

Consequently C_A and C_V can be considered real. After these simplifications, the interacting Hamiltonian is of the form

$$H = \sum_{x=V,A} C_x \langle p | O_x | n \rangle \langle e | O_x (1+\gamma_5) | \nu \rangle \delta(\vec{R} - \vec{r}) + h.c. \qquad (1\text{-}5)$$

The transition probability for a beta decay can be obtained by calculating the interacting matrix, H. The evaluation of the leptonic matrix elements $\langle e | O_x (1+\gamma_5) | \nu \rangle$ is possible since the Dirac equation for the electron and the neutrino can be solved with good accuracy. The quantities represented by $\langle p | O_x | n \rangle$ are independent of the electron and neutrino variables and are called the beta decay nuclear matrix elements. In order to evaluate the matrix elements for the various operators, O_x, one must have wave functions describing the nucleus before and after the emission process. These must be obtained by adopting a particular nuclear model.

All of the theoretical expressions for the experimental observables of beta decay, such as the transition probability, the spectrum shape correction factor, and the beta-gamma angular correlation, depend upon the nuclear matrix elements. Therefore, a knowledge of these matrix elements is, indeed, very desirable. The agreement between theoretical predictions and experimental results will, in turn, determine the feasibility of the nuclear model.

History of Nuclear Models

The earliest, working nuclear model, the liquid-drop model, was proposed by Niels Bohr[8] in 1936. This model assumes an irrotational hydrodynamic flow and describes the collective excitation in the form of

rotations and surface vibrations. The comparison of the nucleus with a liquid drop has met with considerable success in the theory of nuclear reaction and of nuclear fission[9]. However, since the number of nucleons is of little importance in this model, it can hardly describe quantitatively many experimental evidences such as magic numbers, nuclear spins, etc.

In 1949, the shell model was proposed independently by M. G. Mayer and O. Haxel et al.[10]. This model assumes that each nucleon, like an electron in an atom, moves independently in an average, spherically symmetric potential well, generated by its interaction with all the other nucleons. The potential function is assumed to be

$$V = V(r) - W(r)\,\vec{l}\cdot\vec{s}\,, \qquad (1\text{-}6)$$

where r is the distance from the center of force coinciding with the center of mass of the nucleus, and the term $\vec{l}\cdot\vec{s}$ is the hypothetical spin-orbit coupling. The potential $V(r)$ can be an isotropic harmonic oscillator, an infinite square well, or an intermediate potential well between those two. The spin-orbit term is introduced to reproduce the magic numbers. The harmonic oscillator, for instance, can only provide correctly the first three magic numbers (2,8,20). It is quite natural to assume the existence of the spin-orbit coupling since it is well known that the two-body nuclear force depends on the spin and the requirement that the Hamiltonian be an invariant can only lead to the spin-orbit type.

In the extreme independent particle model, nucleons fill the single-particle orbits according to the Pauli exclusion principle. The nucleons in time-reversed orbits are paired off to zero angular momentum, and the ground-state spin of an odd-mass nucleus is equal to the angular momentum of the last odd nucleon.

The spherical shell model covers a wide range of nuclear data, particularly for nuclei with proton- and neutron-numbers near the magic numbers. However, for heavy nuclei far away from closed-shell, the following discrepancies are observed[11]:

1). The static electric quadrupole moments are consistently large, by factors of 20 to 100, compared with those predicted by the shell model.

2). The E2 transition rates are at least 100 times greater than that the shell model estimates.

3). The rotational energy spectra can only result from a non-spherical system, similar to a diatomic molecule.

These facts can only be accounted for by some sort of cooperative behavior of nucleons, such as the behavior postulated for the liquid-drop model. However, the assumptions for these two models are just opposite.

The unified model proposed by A. Bohr and B. R. Mottelson[12] provides a synthesis of the apparently contradictory independent-particle and collective properties of the nuclear motion. In order to describe simultaneously the individual properties of the nucleons and the collective behavior of the nucleus, it is necessary that the Hamiltonian for the system embody both collective and intrinsic variables. Therefore it is assumed the Hamiltonian may be expressed as

$$H = H_{collective} + H_{particle} + H_{interaction} . \qquad (1\text{-}7)$$

$H_{collective}$, the Hamiltonian for the collective motion, depends only upon a set of collective variables (α). In order to obtain a model which is mathematically solvable, (α) must be chosen in such a way that it manifests the orientation of the nucleus in space as well as its shape. Thus, by a suitable choice of (α), it is possible to separate the collective motion into a rotational and a vibrational motion. $H_{collective}$ may then be written as

$$H_{collective} = H_{rotation} + H_{vibration} . \qquad (1\text{-}8)$$

$H_{rotation}$ can be a function of only 3 variables. The remaining variables are contained in $H_{vibration}$. The number of variables in $H_{variation}$ will depend on the approximations that are made.

$H_{particle}$ is the Hamiltonian for the particle motion. Since the orientation of the nucleus is already specified, $H_{particle}$ is a function of a set of intrinsic variables (ξ') related to nuclear axes. As the name of the model itself indicates, $H_{particle}$ is expected to be similar to the Hamiltonian of the shell model.

Finally, $H_{interaction}$ represents the interaction between the collective and particle motion, and therefore depends on both (α) and (ξ'). The explicit form of $H_{interaction}$ depends on more assumptions to make the problem soluble.

If it is assumed that the nuclear shape changes slowly compared with the particle motion, the nuclear wave function can be written as

$$\Psi \sim \Phi_{vib} D_{rot} X_{int} . \qquad (1\text{-}9)$$

Φ_{vib} represents the vibrations of the nucleus around its equilibrium shape, D_{rot} the rotational motion of the nucleus, and X_{int} the intrinsic motion of individual nucleons.

Interest in the present problem will be focused on strongly deformed nuclei. Since the nuclear shape lasts for a sufficiently long time, the nucleus is assumed to be in the vibrational ground state. Therefore, Φ_{vib} plays no role in the calculation at all and will be suppressed hereafter.

After the formulation of the theory of superconductivity, nuclear theory came to a new stage. In 1958, A. Bohr et al.[13] pointed out the energy gap in the excitation spectra of certain nuclei, in analogy with the excitation spectra of the superconducting metallic state. This suggests the notion of "pairing" between nucleons, similar to a Cooper-pair in a superconductor, and has been the subject of much study ever since[14].

Purpose of this Research

Theoretical expressions for the nuclear matrix elements in beta decay have been derived by Rose and Osborn[15] for the shell model, and by Bogdan[16] for the Nilsson single-particle model[17]. Throughout his work, Bogdan describes the intrinsic states, X_Ω, of the nucleus as single-particle Nilsson states. Bogdan calculates the nuclear matrix elements for the beta decay of odd-odd nuclei by assuming X_Ω is simply the appropriate Nilsson state for the transforming nucleon. He does not consider any contribution to X_Ω from the non-transforming unpaired nucleon.

In Nilsson's original paper[17], the general expressions for the nuclear matrix elements for electromagnetic transitions were given for odd-mass nuclei. The classification of beta and gamma transitions between intrinsic states in deformed even-mass nuclei was discussed by Gallagher[18], using two-particle intrinsic wave functions. In this case, X_Ω becomes the product of two Nilsson single-particle wave functions. For instance, for an odd-odd nucleus it is assumed that the last odd proton and odd neutron move independently in Nilsson states and that X_Ω is the product of the single-particle states appropriate for that Z and N.

In this investigation, the general expressions for nuclear matrix elements in beta decay, together with their selection rules, will be given by making use of the nuclear wave functions of the unified nuclear model. These wave functions have the form

$$\Psi \sim D_{rot}\, \chi_{int} \quad , \tag{1-10}$$

where D_{rot} and X_{int} are the rotational and intrinsic wave functions, respectively. The intrinsic wave function will be constructed from the single-particle wave function given by Nilsson[17]. Nilsson obtained these wave functions by assuming that the single-particle Hamiltonian has the form of an anisotropic harmonic oscillator with spin-orbit coupling. The beta decay operators are written in terms of irreducible spherical tensors as defined by Rose[15], and will be discussed in some detail in Chapter III.

In the present calculations, one- and two-particle intrinsic wave functions will be used for odd- and even-mass nuclei, respectively. The

two-particle intrinsic wave functions will have the general form given by Gallagher. An essential difference between Bogdan's work and the present work is the use of the two-particle intrinsic wave functions. The basic difference between the use of one- and two-particle wave function is the selection rules. As was pointed out by Gallagher[18], when the two-particle intrinsic wave function is used, the selection rules also depend on the coupling of the two particles. Moreover, the total nuclear spin J of odd-odd nuclei will be determined by the coupling rules for the projection quantum number Ω_i of the last two nucleons. These coupling rules are given by Gallagher and Moszkowski[19].

Since the calculation is made for a tensor of an arbitrary rank, the matrix elements so obtained may be applied to any degree of forbiddenness. The derived theoretical expressions are generally applicable to beta decays in the atomic mass region for which the Nilsson model is a useful description (A 25, $150 < A < 190$, $A > 200$). However there are certain transitions for which the expressions are not applicable due to the nature of the initial or final nuclear state. Since the nucleus is assumed to be in the vibrational ground state the results are not applicable to decays involving states that are collective vibrational excitations. Also, there are certain transitions for which the one-particle intrinsic wave functions for odd-mass nuclei and the two-particle intrinsic wave functions for even-mass nuclei will not be appropriate. For example, there are some beta decays of odd-odd nuclei where the decaying neutron is interpreted as being one of the paired neutrons instead of the last odd neutron. This situation can not be described with two-particle wave functions. Since the electromagnetic

transition, as far as mathematics is concerned, can be considered as a special case of beta decay, on many occasions, the derivation can also be applied to electromagnetic transitions.

As an application of the derived theoretical equations, numerical results have been obtained for the nuclear matrix elements for beta decays in Tm^{170} and Re^{186}. Both transitions are from the ground state of an odd-odd nucleus to the first excited state of an even-even nucleus. For both cases, two-particle intrinsic wave functions are appropriate. The theoretically predicted matrix elements are compared with experimental values.

CHAPTER II

THE BOHR-MOTTELSON-NILSSON UNIFIED NUCLEAR MODEL

For nuclei in the region between two magic numbers, the description of the nucleus as a deformed body is found to be particularly useful. Furthermore, although the nucleons play an essential role in this deformed body, they do not lose their individual entity. A realistic nuclear model, therefore, must include both collective and individual aspects. The mathematical details for this unified nuclear model were worked out by Bohr, Mottelson and Nilsson.

As was briefly discussed in the introduction, the Hamiltonian for the deformed nucleus is assumed to be

$$H=H_{collective}+H_{particle}+H_{interaction} . \tag{2-1}$$

The collective dynamical variables $\alpha_{\lambda\mu}$ are usually defined by an expansion of the nuclear surface in terms of spherical harmonics, $Y_{\lambda\mu}$. This gives

$$R(\vartheta,\varphi)= R_0 \left[1+\sum_{\lambda,\mu}\alpha_{\lambda\mu} Y_{\lambda\mu}(\vartheta,\varphi)\right] , \tag{2-2}$$

where R_0 is the radius of the nucleus in its spherical equilibrium shape, and ϑ and φ are polar angles with respect to arbitrary axes. The Hamiltonian for the collective motion can then be written in terms of $\alpha_{\lambda\mu}$ and its first time-derivative. In his original paper, Bohr[12] considered only the lowest approximation with $\lambda=2$ which corresponds to the

quadrupole deformation. $\lambda = 1$ corresponds to a dipole oscillation. With a simple argument on parity, it can be shown that the nuclear dipole moment equals zero. Thus, in the first approximation, there are five parameters, $\alpha_{2\mu}$, that describe the collective nuclear motion.

If the principal axes of the nucleus are chosen as a system of axes fixed in the nuclear frame and related to the space-fixed coordinate system by three Euler angles θ_i, the parameters $\alpha_{2\mu}$ will transform according to (Appendix I)

$$\alpha_{2\mu} = \sum_{\nu} a_{2\nu} D^{*2}_{\mu\nu}(\theta_i) \quad . \tag{2-3}$$

Since the body axes are principal axes, it follows that

$$a_{21} = a_{2,-1} = 0 \quad , \quad \text{and}$$

$$a_{22} = a_{2,-2} \quad .$$

The new five variables are now a_{20}, a_{22} and $\theta_i (i=1,2,3)$. For convenience, two variables β and γ are introduced and defined by

$$a_{20} = \beta \cos\gamma \quad , \quad \text{and} \tag{2-4a}$$

$$a_{22} = \frac{1}{\sqrt{2}} \beta \sin\gamma \quad . \tag{2-4b}$$

The collective Hamiltonian can then be separated into a vibrational and a rotational part. A formal mathematical development can be carried out from this point. However, due to additional assumptions, there is a somewhat simpler way to describe strongly deformed nuclei. This will be discussed later.

The collective motion of the nucleus and the motion of individual nucleons are basically related. $H_{interaction}$ can be embodied in $H_{particle}$, provided the potential in $H_{particle}$ is allowed to depend on the parameters characterizing the shape of the nucleus. For strongly deformed nuclei, the approach of writing $H_{particle}$ and $H_{interaction}$ together is very convenient[17]. As it turns out, β is a measure of the total deformation and is chosen as a fixed empirical parameter for each nucleus, γ is a constant angle (see Section II of this chapter). For a fixed β and γ, the collective motion is strictly described by $H_{rotation}$ instead of a complicated collective Hamiltonian. These assumptions will be used as a starting point for the unified model described in Section I of this chapter. The intrinsic structure of the nucleus, together with the Nilsson potential will be considered in Section II.

Rotational Nuclear Model

For strongly deformed nuclei, the nucleus is assumed to have a permanent non-spherical shape. This assumption can be justified since the nuclear shape lasts for a sufficiently long time compared with the motion of individual nucleons. For further simplicity the nucleus is assumed to be spheroidal. As a consequence, the nucleons will move in an average non-spherical field. The appearance of the rotation energy spectra is a natural result of the assumed non-spherical shape. Since the rotational motion is solely responsible for the collective motion, the unified model can be justifiably called the rotational model in this special case.

In order to describe the rotational motion of the nucleus, the most

suitable dynamical variables probably are the Euler angles $\theta_i(\alpha,\beta,\gamma)$ which specify the orientation of the nucleus with respect to a system of axes , Oxyz , fixed in the laboratory frame. For convenience, the principal axes , Ox'y'z', are chosen as the system of coordinates fixed in the nucleus. All the other variables, as well as their conjugate momenta, describing the intrinsic motion of individual nucleons will be globally called (ξ'). The description of the nuclear motion is by no means complete with these variables. Another set of empirical parameters are still needed to specify the shape of the nucleus, i.e., the degree of its departure from the spherical shape. These parameters can be adjusted to give best results for the energy spectra or for the quadrupole moment. The Hamiltonian of the system is assumed to be

$$H = H_{rot}(\theta_i) + H_{int}(\xi') + H_{coupl} , \qquad (2\text{-}5)$$

where $H_{rot}(\theta_i)$ is the kinetic energy operator for the rotational motion, $H_{int}(\xi')$ the intrinsic Hamiltonian, and H_{coupl} represents the coupling of the rotational and the intrinsic motion.

Eigenfunctions of the Hamiltonian

Despite the simplifications already made, it is not easy to obtain the eigenfunctions for the Hamiltonian (2-5). Another basic assumption is thus introduced and called "the adiabatic assumption". This assumption says: the single-particle states are not affected by the excitation of the collective motion. Mathematically speaking it means that the eigenfunctions of the Hamiltonian (2-5) can be separated into collective and intrinsic eigenfunctions. Therefore,

$$\Psi \sim D(\theta_i)\, X(\xi') , \tag{2-6}$$

where $D(\theta_i)$ is an eigenfunction of H_{rot}, and $X(\xi')$ is an eigenfunction of $H_{int}(\xi')$. If the adiabatic assumption is to be valid, it is necessary that H_{coupl} be so small compared with H_{rot} and H_{int} that it can be neglected.

Now let $\vec{R}$ be the angular momentum due to the collective motion and $\vec{j}$ the intrinsic angular momentum, associated with the motion of individual nucleons. The total angular momentum of the nucleus is then given by

$$\vec{J} = \vec{R} + \vec{j} . \tag{2-7}$$

It is necessary that the eigenfunctions of H also be those of J^2, J_z and $J_{z'}$, (J_z and $J_{z'}$ are the projections of $\vec{J}$ on Oz, the space fixed axis, and Oz', the intrinsic-nuclear axis, respectively).

The details of H_{int} and its eigenfunctions will be left out until the next section. Here, it is simply noted that j is not a good quantum number since the potential is not spherical. However, $j_{z'}$ is a good quantum number because the nucleus is assumed to be axially symmetric. It follows that

$$j_{z'}\, X^{\tau}_{\Omega}(\xi') = \Omega\, X^{\tau}_{\Omega}(\xi') , \tag{2-8}$$

where τ denotes all other quantum numbers specifying the intrinsic state. τ will be dropped hereafter.

If the nucleus is treated as a rigid body, H_{rot} will have the form

$$H_{rot} = \frac{R^2_{x'}}{2I_{x'}} + \frac{R^2_{y'}}{2I_{y'}} + \frac{R^2_{z'}}{2I_{z'}} \tag{2-9}$$

where the I's are the principal moments of inertial. The I's depend on the empirical parameters which characterize the shape of the nucleus. For an axially symmetric nucleus, two principal moments of inertia are equal, so that

$$I_{x'} = I_{y'} = I \,. \tag{2-10}$$

Combining Equations (2-10) and (2-7) with Equation (2-9) gives:

$$H_{rot} = \frac{1}{2I}\left(J^2 + j^2 - 2\vec{J}\cdot\vec{j}\right) + \left(\frac{1}{2I_{z'}} - \frac{1}{2I}\right)\left(J_{z'} - j_{z'}\right)^2 . \tag{2-11}$$

In H_{rot} the term j^2 is entirely a function of the intrinsic motion, and therefore can be absorbed in a redefined Hamiltonian, $H_{int}(\xi')$. Moreover, if the term $\vec{J}\cdot\vec{j}$ can be neglected, then D^J_{MK} is precisely the eigenfunction of the rotational Hamiltonian (2-11) and satisfies

$$J^2 D^J_{MK}(\theta_i) = J(J+1) D^J_{MK}(\theta_i) \,, \tag{2-12a}$$

$$J_z D^J_{MK}(\theta_i) = M D^J_{MK}(\theta_i) \,, \tag{2-12b}$$

$$J_{z'} D^J_{MK}(\theta_i) = K D^J_{MK}(\theta_i) \,. \tag{2-12c}$$

The properties of the eigenfunctions $D^J_{MK}(\theta_i)$ are discussed in Appendix I.

The normalized eigenfunction of the total Hamiltonian (2-5) is

$$\Psi(JMK\Omega) = \left(\frac{2J+1}{8\pi^2}\right)^{1/2} D^J_{MK}(\theta_i)\, \chi_\Omega(\xi') \,. \tag{2-13}$$

The coupling of the angular momenta associated with the wave function is shown in Figure 1.

Consideration of Neglected Term in the Hamiltonian

Before looking into more details of $\psi(JMK\Omega)$, it is instructive to discuss the effect on $\psi(JMK\Omega)$ of the neglected term $\vec{J}\cdot\vec{j}$. This operator can be written

$$\vec{J}\cdot\vec{j} = J_{z'}j_{z'} + \frac{1}{2}\left(J_+ j_- + J_- j_+\right), \qquad (2\text{-}14)$$

where

$$J_\pm = J_{x'} \pm i J_{y'} \qquad \text{and} \qquad j_\pm = j_{x'} \pm i j_{y'} . \qquad (2\text{-}15)$$

$\psi(JMK\Omega)$ is obviously an eigenfunction of $J_{z'}j_{z'}$ with eigenvalue $K\Omega$. This operator $J_{z'}j_{z'}$ only shifts the energies of all states by the same amount, therefore does not affect the relative spacings. In order to study the next two terms, $J_\pm j_\mp$, $X_\Omega(\xi')$ is expanded into the eigenfunctions $X_{j\Omega}(\xi')$ of j^2 (since X_Ω is not an eigenfunction of j^2). Therefore,

$$X_\Omega = \sum_j c_j X_{j\Omega} . \qquad (2\text{-}16)$$

Then

$$\Psi(JMK\Omega) = \left(\frac{2J+1}{8\pi^2}\right)^{\frac{1}{2}} \sum_j c_j D^J_{MK} X_{j\Omega} , \qquad (2\text{-}17)$$

and (see Appendix I)

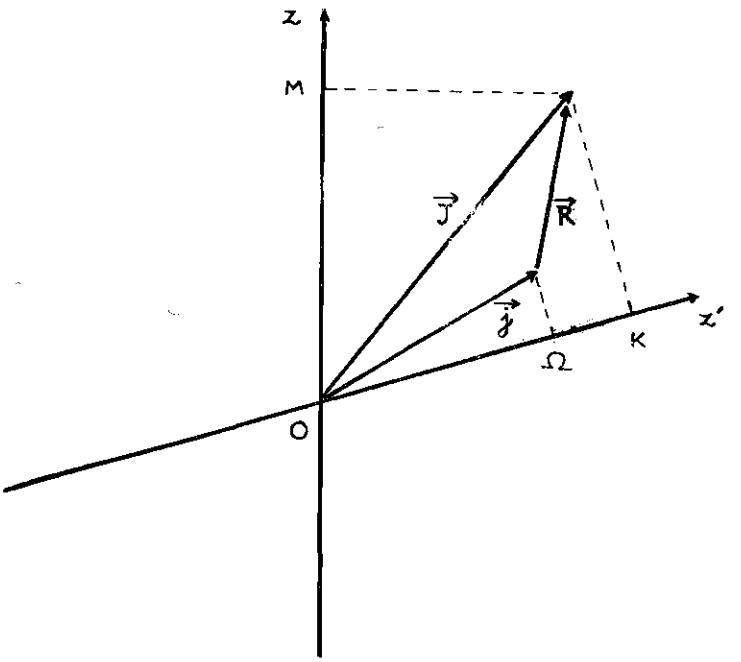

Figure 1. Coupling of Angular Momenta of a Deformed Nucleus.

$\vec{J}$ represents the total nuclear angular momentum, $\vec{j}$ is the angular momentum due to the intrinsic motion of the nucleons, and $\vec{R}$ is the angular momentum generated by the collective motion of the nucleus. Oz and Oz' are the space fixed axis and the nuclear symmetry axis, respectively.

$$J_{\pm} j_{\mp} D^{J}_{MK} \chi_{j\Omega} = \left[(J \pm K)(J \mp K + 1)(j \pm \Omega)(j \mp \Omega + 1)\right]^{1/2} D^{J}_{M,K\mp1} \chi_{j\Omega\mp1} \quad (2\text{-}18)$$

If all states with the same intrinsic wave function χ_{Ω} but different values of J are called a rotational band, it is clear that the operators $J_{\pm}j_{\mp}$ mix different rotational bands with Ω-values differing by 1. Then Ω is no longer a good quantum number as previously assumed. However, this band-mixing is nelgigibly small, compared with the intrinsic excitation energy (diagonal terms)[11]. This is not the case however when $\Omega=K=1/2$. Using the correct symmetrized wave function, it can be shown that $J_{\pm}j_{\mp}$ contributes an additional non-vanishing diagonal term[17],

$$\langle J_+ j_- + J_- j_+ \rangle = a(-1)^{J+\frac{1}{2}}(J+\tfrac{1}{2}) \quad , \qquad (2\text{-}19)$$

where "a", the decoupling parameter, is given by

$$a = \sum_{j} (-1)^{j-\frac{1}{2}} (j+\tfrac{1}{2}) \left| C_{j\frac{1}{2}} \right|^2 \quad . \qquad (2\text{-}20)$$

Hereafter, only the case where Ω is a good quantum number will be considered.

Symmetry of the Nuclear Wave Function

From the beginning, the nucleus is assumed to have a spheroidal shape. Thus, it is required that the nuclear wave function be invariant under a rotation about the symmetry axis through an arbitrary angle φ and also under a rotation through an angle π about an axis normal to the symmetry axis and passing through the center.

The first rotation is denoted by $R(\hat{z}',\varphi)$ where Oz' is chosen as the

symmetry axis. Its effects on $D^J_{MK}(\theta_i)$ and $X_\Omega(\xi')$ will now be considered. Since D^J_{MK} is an eigenfunction of J^2, under the rotation $R(\hat{z}',\varphi)$ it transforms according to (Appendix I)

$$R(\hat{z}',\varphi)\, D^J_{MK} = \sum_{K'} D^J_{KK'}(0\,0\,\varphi)\, D^J_{MK'} = \sum_{K'} e^{i\varphi K'} \delta_{KK'} D^J_{MK'} , \qquad (2\text{-}21)$$

i.e.
$$R(\hat{z}',\varphi)\, D^J_{MK} = e^{i\varphi K} D^J_{MK} \qquad (2\text{-}22)$$

Since $X_\Omega(\xi')$ is not an eigenfunction of j^2, it is first expanded in terms of $X_{j\Omega}$ as in Equation (2-16). Then under the rotation $R(\hat{z}',\varphi)$ one has

$$R(\hat{z}',\varphi)\, X_{j\Omega} = \sum_{\Omega'} D^{*j}_{\Omega'\Omega}(0\,0\,\varphi)\, X_{j\Omega'} = \sum_{\Omega'} e^{-i\varphi\Omega} \delta_{\Omega\Omega'} X_{j\Omega'} , \qquad (2\text{-}23)$$

i.e.
$$R(\hat{z}',\varphi)\, X_\Omega = e^{-i\varphi\Omega} X_\Omega \, . \qquad (2\text{-}24)$$

Note the star on $D^j_{\Omega'\Omega}$ in Equation (2-23). The difference between the two transformations, Equations (2-21) and (2-23), is due to different reference frames (Appendix I). Finally $\psi(JMK\Omega)$ transforms according to

$$R(\hat{z}',\varphi)\Psi(JMK\Omega) = e^{i(K-\Omega)\varphi} \Psi(JMK\Omega) . \qquad (2\text{-}25)$$

If the function $\psi(JMK\Omega)$ is to be invariant under this rotation, it is necessary that

$$K = \Omega \, . \qquad (2\text{-}26)$$

The second rotation, denoted by $R(\hat{y}',\pi)$, can be studied in an analogous manner. Hence,

$$R(\hat{y}',\pi)\, D^{J}_{MK} = \sum_{K'} D^{J}_{KK'}(0\pi 0)\, D^{J}_{MK'} = \sum_{K'} (-1)^{J-K} \delta_{K,-K'} D^{J}_{MK'} , \quad (2\text{-}27)$$

i.e.
$$R(\hat{y}',\pi)\, D^{J}_{MK} = (-1)^{J-K} D^{J}_{M-K} . \quad (2\text{-}28)$$

$$R(\hat{y}',\pi)\, \chi_{j\Omega} = \sum_{\Omega'} D^{*j}_{\Omega'\Omega}(0\pi 0)\, \chi_{j\Omega'} = \sum_{\Omega'} (-1)^{j-\Omega} \delta_{\Omega,-\Omega'} \chi_{j\Omega'} , \quad (2\text{-}29)$$

or
$$R(\hat{y}',\pi)\, \chi_{j\Omega} = (-1)^{j-\Omega} \chi_{j-\Omega} , \quad (2\text{-}30)$$

i.e.
$$R(\hat{y}',\pi)\, \chi_{\Omega} = (-1)^{j-\Omega} \chi_{-\Omega} . \quad (2\text{-}31)$$

In Equation (2-31), $(-1)^j$ must be understood as an operator acting on each component $X_{j\Omega}$ in the expansion. Thus,

$$(-1)^{j} \chi_{\Omega} = \sum_{j} (-1)^{j} C_j \chi_{j-\Omega} . \quad (2\text{-}32)$$

Since $j-\Omega$ is always an integer, it follows that

$$R(\hat{y}',\pi)\Psi(JMK\Omega) = (-1)^{J-j} \Psi(JM-K,-\Omega) . \quad (2\text{-}33)$$

Provided $K \neq 0$, it can be seen that $\psi(JMK)$ is not invariant under this rotation. However, the linear combination $\psi(JMK) + R(\hat{y}',\pi)\psi(JMK)$ is invariant. The normalized wave function is then given by

$$|JMK\rangle = \left(\frac{2J+1}{16\pi^2}\right)^{1/2} \left\{ D^{J}_{MK}(\theta_i)\, \chi_{\Omega}(\xi') + (-1)^{J-j} D^{J}_{M-K}(\theta_i)\, \chi_{-\Omega}(\xi') \right\} . \quad (2\text{-}34)$$

Since $K \leqslant J$, provided $K \neq 0$, the sequence

$$J=K,\ K+1,\ K+2,\ \ldots \quad (2\text{-}35)$$

constitutes the rotational band with the ground state K.

If $K = 0$ a different situation exists. Equation (2-23) then becomes

$$R(\hat{y}',\pi)\Psi(JMK=0) = (-1)^{J-j}\Psi(JMK=0), \tag{2-36}$$

$\psi(JMO)$ itself now has definite parity. The second term in the linear combination (2-34) either contributes nothing new or cancels off the first term. Therefore, the normalized wave function for the case $K = 0$ will be taken as

$$|JMK=0\rangle = \left(\frac{2J+1}{16\pi^2}\right)^{1/2} D^{J}_{M0}(\theta_i)\,\chi_0(\xi'). \tag{2-37}$$

Intrinsic Structure of the Nucleus

At the present time, it is not possible to describe the intrinsic structure of the nucleus in a precise manner since the exact nature of the nuclear force is not known. The practical way to solve the problem then consists of postulating a certain average potential in which the nucleons move. In so doing, the problem bears a strong similarity with the well-known atomic problem. For a deformed nucleus, some form of an anisotropic harmonic oscillator potential is generally used.

To represent the interaction of one nucleon with the nuclear field, Nilsson[17] assumed an intrinsic Hamiltonian of the form

$$H = H_0 + C\vec{l}\cdot\vec{s} + Dl^2, \tag{2-38}$$

where

$$H_o = -\frac{1}{2M}\nabla'^2 + \frac{1}{2}M(\omega_x^2 x'^2 + \omega_y^2 y'^2 + \omega_z^2 z'^2) \,. \qquad (2\text{-}39)$$

The $\vec{l}\cdot\vec{s}$-term is the spin-orbit interaction as in the shell model. The l^2-term serves to depress the high angular momentum states. The parameters C and D are chosen in such a way that the observed single-particle level structure near closed shells must be reproduced at zero deformation. The ω's depend on a single deformation parameter δ in the following manner

$$\omega_x^2 = \omega_y^2 = \omega_o^2(1 + \tfrac{2}{3}\delta) \,, \qquad (2\text{-}40a)$$

$$\omega_z^2 = \omega_o^2(1 - \tfrac{4}{3}\delta) \,. \qquad (2\text{-}40b)$$

The condition of constant nuclear volume leads to

$$\omega_x \omega_y \omega_z = \text{constant} \,. \qquad (2\text{-}41)$$

If Equation (2-41) is combined with Equations (2-40a) and (2-40b), ω_o is then given by

$$\omega_o(\delta) = \omega_o(o)\left(1 - \tfrac{4}{3}\delta^2 - \tfrac{27}{16}\delta^3\right)^{-1/6} \,, \qquad (2\text{-}42)$$

where $\omega_o(0)$ is the value of $\omega_o(\delta)$ when $\delta=0$.

Now it is convenient to introduce the new coordinates

$$\xi = (M\omega_o)^{1/2} x' \,, \quad \eta = (M\omega_o)^{1/2} y' \,, \quad \zeta = (M\omega_o)^{1/2} z' \,. \qquad (2\text{-}43)$$

H_o then can be written as

$$H_o = \overset{o}{H}_o + H_\delta \quad , \tag{2-44}$$

where

$$H_o = \tfrac{1}{2}\omega_o(-\nabla^2 + \rho^2) \quad , \tag{2-45}$$

$$H_\delta = -\tfrac{4}{3}\sqrt{\tfrac{\pi}{5}}\,\omega_o \rho^2 \delta\, Y_{20} \quad , \tag{2-46}$$

and

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2 \quad . \tag{2-47}$$

It turns out that for Nilsson's potential, the quantities β and γ defined by Equations (2-4a) and (2-4b) become

$$\gamma = 0 \quad , \tag{2-48a}$$

$$\beta = \tfrac{4}{3}\sqrt{\tfrac{\pi}{5}}\,\delta \quad . \tag{2-48b}$$

Since $\overset{o}{H}_o$, l^2, l_z and s_z commute with one another, Nilsson chose a representation in which $\overset{o}{H}$, l^2, l_z and s_z are diagonal. The basic vectors are then $|N\ell\Lambda\Sigma\rangle$ and satisfy

$$\overset{o}{H}_o | \; \rangle = (N + \tfrac{3}{2})\omega_o | \; \rangle \quad ,^{*} \tag{2-49a}$$

$$\ell^2 | \; \rangle = \ell(\ell+1) | \; \rangle \quad , \tag{2-49b}$$

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* Details of Equation (2-49a) can be found in Appendix II.

$$\ell_z | \rangle = \Lambda | \rangle , \tag{2-49c}$$

$$s_z | \rangle = \Sigma | \rangle . \tag{2-49d}$$

The Hamiltonian (2-38) is now written

$$H = \overset{\circ}{H}_0 + H_\delta + C \vec{\ell}.\vec{s} + D \ell^2 . \tag{2-50}$$

In the chosen representation, $\overset{\circ}{H}_0$ and l^2 are diagonal a priori. However, H_δ and $\vec{l}.\vec{s}$ are not diagonal. In fact, H_δ is diagonal in Λ and Σ but couples l with $l \pm 2$ and N with $N \pm 2$, while $\vec{l}.\vec{s}$ is diagonal in N and l but couples Λ with $\Lambda \pm 1$ and Σ with $\Sigma \pm 1$.

The full energy matrix can be obtained without much difficulty. However, the diagonalization of the infinite matrix with terms coupling N with $N \pm 2$ is very involved. Fortunately, these off-diagonal terms are much smaller than the diagonal ones[17] and, consequently, are neglected. N is then also a good quantum number. Furthermore, although l_z and s_z do not commute with H, $j_z = \ell_z + s_z$ does commute with H. The corresponding quantum number is denoted by Ω. Thus, for each set of N and Ω, there is one matrix to diagonalize. The vectors with $\Omega = \Lambda + \Sigma$ are used as basic vectors. The diagonalization yields a set of single-particle eigenvalues and corresponding eigenfunctions, depending on the deformation parameters.

Since $\overset{\circ}{H}_0$ is diagonal in the chosen representation, Nilsson defined a new operator R by

$$H - H_0 = \varkappa \omega_0(0) R , \tag{2-51}$$

where

$$R = \eta U - 2\vec{\ell}\cdot\vec{s} - \mu\vec{\ell}^2 , \tag{2-52a}$$

$$U = -\frac{1}{3}\sqrt{\frac{1}{5}}\,r^2 Y_{20} , \tag{2-52b}$$

$$\varkappa = -\frac{C}{2\omega_0(0)} , \tag{2-53}$$

$$\mu = \frac{D}{C} \tag{2-54}$$

$$\eta = \frac{\delta}{\varkappa}\,\frac{\omega_0(\delta)}{\omega_0(0)} \tag{2-55}$$

In this diagonalization, R is treated as a function of η (or δ) and diagonalized for a sequence of η- (or δ-) values. μ is chosen depending on N . $\varkappa$ is taken to be equal to 0.05.

Nilsson gives an extensive tabulation of eigenvalues and eigenfunctions (the values of $a_{\ell\Lambda}$ in the expansion $\Sigma a_{\ell\Lambda}|N\ell\Lambda\Sigma\rangle$) up to N=6. Nilsson also plots the energy levels as function the parameter η .

The results obtained by Nilsson are remarkable. However, in view of the three parameters besides $\omega_0(0)$ which can be chosen quite freely, the validity of the theory is rather arbitrary. Nevertheless, for the time being, Nilsson's works serve, as M. Baranger* put it, as the "bible of experimental nuclear physicists".

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* M. Baranger, in 1962 Cargese Lectures in Theoretical Physics, edited by M. Levy, p. V7, Benjamin Inc., New York (1963).

CHAPTER III

SPHERICAL TENSOR OPERATORS IN BETA AND ELECTROMAGNETIC TRANSITIONS

As a result of the requirement of Lorentz invariance in Dirac theory, at most five different types of interaction can be constructed and listed, conventionally, S, V, T, A and $P^{(2)}$. These interactions transform covariantly and can fully describe the beta decay process. The theoretical calculation of the transition probability and the other experimental observables of beta decay requires the evaluation of leptonic and nuclear matrix elements for these operators. The leptonic matrix elements can be obtained with accuracy, since the Dirac equation for light particles can be solved with many refinements. The nuclear matrix elements can only be obtained from a particular model of the nucleus.

The commonly used definitions for the operators and their corresponding matrix elements were first presented by Konopinski and Uhlenbeck[20]. Kotani and Ross[21] have presented theoretical expressions for the various experimental observables of first-forbidden beta decay in terms of nuclear matrix elements as defined by Konopinski and Uhlenbeck. Rose and Osborn[15] have discussed the representation of beta decay operators in terms of spherical tensors. They also have presented the relation between the operators and corresponding matrix elements in the spherical tensor notation and in the customary notation of Konopinski and Uhlenbeck. In the present work it was found convenient

to derive theoretical expressions for the matrix elements in terms of the notation and definitions of Rose and Osborn. In this chapter various aspects of the spherical tensor representation of beta decay operators are discussed. The discussion closely follows that presented by Rose and Osborn.

Rose and Osborn have reformulated the theory of forbidden beta decay in a representation in which only even Dirac operators occur in the nuclear space. The reformulation proceeds by means of a Foldy-Wouthuysen transformation and implies that nonrelativistic nuclear wave functions may be used for the description of beta transitions. In the new formulation, the following substitutions are made for the odd operators:

$$\vec{\alpha} \rightarrow -\frac{1}{M}\vec{p} \quad , \qquad i\beta\vec{\alpha} \rightarrow \frac{1}{M}\vec{\sigma}\times\vec{p} \quad , \qquad \gamma_5 \rightarrow \frac{1}{M}\vec{\sigma}\cdot\vec{p} \quad ,$$

where M is the nucleon mass and $\vec{p} = -i\vec{\nabla}$.

The various beta decay operators are expressed by Rose and Osborn in terms of spherical tensors defined by

$$T_{\lambda L m}(\vec{r},\vec{\omega}) = \sum_{m'} \langle 1\, L\, -m'\ \ m'+m \mid \lambda\ m\rangle\, \mathcal{Y}_{L\ m'+m}(\vec{r})\, \mathcal{Y}_{1,-m'}(\vec{\omega}), \qquad (3\text{-}1)$$

where $\langle 1\, L\, -m'\ m'+m \mid \lambda\ m\rangle$ is the C-coefficient (Appendix I). $\mathcal{Y}_{L,m'+m}(\vec{r})$ is the solid spherical harmonics of degree L, and

$$\mathcal{Y}_{1m}(\vec{\omega}) = \sqrt{\frac{3}{4\pi}}\,\omega_m \quad \text{with} \quad \begin{cases} \omega_1 = -\frac{1}{\sqrt{2}}(\omega_x + i\omega_y) \\ \omega_0 = \omega_z \\ \omega_{-1} = \frac{1}{\sqrt{2}}(\omega_x - i\omega_y) \end{cases} . \qquad (3\text{-}2)$$

In this notation, the tensor operator $T_{\lambda Lm}(\vec{r},\vec{\omega})$ is of rank λ and of parity π, given by

$$\pi = \pi(L)\,\pi(\omega) = (-1)^{L}\pi(\omega) \,. \tag{3-3}$$

For beta decay of arbitrary forbiddenness order all of the operators fall into five types:

Type I: $\mathcal{Y}_{\lambda m}(\vec{r})$, for S-V interactions,

Type II: $T_{\lambda Lm}(\vec{r},\vec{\sigma})$, for A-T-P interactions,

Type III: $T_{\lambda Lm}(\vec{r},\vec{p})$, for V interactions,

Type IV: $\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p}$, for A interactions,

Type V: $T_{\lambda Lm}(\vec{r},\vec{\sigma}\times\vec{p})$, for T interactions.

For any of the five operators listed above, the transition matrix element between the initial nuclear state $|\,i\rangle$ and the final nuclear state $|\,f\rangle$ can be expressed in the form

$$\langle f|\,T_{\lambda Lm}\,|i\rangle = \langle J_i\,\lambda\,M_i\,m\,|\,J_f\,M_f\rangle\langle f\|\,T_{\lambda L}\,\|i\rangle \,.$$

This relationship is the Wigner-Eckart theorem. J_i and M_i are the initial nuclear state quantum numbers while J_f and M_f refer to the final state. The dependence of the matrix element $\langle f|\,T_{\lambda Lm}\,|\,i\rangle$ on the magnetic quantum numbers is contained entirely in the C-coefficient (Appendix I). The quantity $\langle f\|\,T_{\lambda L}\,\|\,i\rangle$ is called the reduced matrix element of the operator. Frequently it is convenient to express certain beta decay equations in terms of the quantity

$$\sum_{m,M_f} |\langle f|\,T_{\lambda Lm}\,|i\rangle|^2 \,.$$

The relationship between this quantity and the corresponding reduced matrix element is easily established in the following manner.

$$\sum_{m,M_f} |\langle f | T_{\lambda L,m} | i \rangle|^2 = \sum_{m,M_f} |\langle J_i \lambda M_i m | J_f M_f \rangle \langle f \| T_{\lambda L} \| i \rangle|^2$$

$$= |\langle f \| T_{\lambda L} \| i \rangle|^2 \sum_{m,M_f} |\langle J_i \lambda M_i m | J_f M_f \rangle|^2 .$$

Thus

$$\sum_{m,M_f} |\langle f | T_{\lambda L m} | i \rangle|^2 = \frac{2J_f+1}{2J_i+1} |\langle f \| T_{\lambda L} \| i \rangle|^2 .$$

In Table 1, the relation between the reduced matrix elements for the above operators and the corresponding reduced matrix elements in the customary notation of Konopinski and Uhlenbeck is presented. The normalizing factor N is defined so that N is the ratio of the entry in the second column to that in the first one. For example:

$$\int \vec{r} = \sqrt{\frac{4\pi}{3}} \langle f \| Y_1(\vec{r}) \| i \rangle .$$

In Table 2 the parity and the degree of forbiddenness of each operator are given.

For beta transitions, only the first four operators are needed. The last operator comes from the tensor interaction, and was ruled out by the current V-A interaction theory. For the sake of mathematical completeness, the matrix elements for this operator are also evaluated.

Table 1. Correspondence Between Spherical Tensor and Cartesian Tensor Notations for Reduced Matrix Elements in Beta Decay of Degree of Forbiddenness $n \leq 2$.

Spherical Notation	Cartesian Notation	N	n	Interaction
$\langle \Vert T_{10}(\vec{r},\vec{\sigma}) \Vert \rangle$	$\int \vec{\sigma}$	$\frac{4\pi}{\sqrt{3}}$	0	T - A
$\langle \Vert Y_{1}(\vec{r}) \Vert \rangle$	$\int \vec{r}$	$\left(\frac{4\pi}{3}\right)^{1/2}$	1	S - V
$\langle \Vert T_{10}(\vec{r},\vec{p}) \Vert \rangle$	$\int \vec{\alpha}$	$\frac{1}{M}\frac{4\pi}{\sqrt{3}}$	1	V
$\langle \Vert T_{11}(\vec{r},\vec{\sigma}) \Vert \rangle$	$\int i\ \vec{\sigma} \times \vec{r}$	$\frac{\sqrt{32\pi}}{3}$	1	T - A
$\langle \Vert T_{21}(\vec{r},\vec{\sigma}) \Vert \rangle$	$\int B_{ij}$	$\frac{8\pi}{3}$	1	T - A
$\langle \Vert \vec{\sigma} \cdot \vec{p} \Vert \rangle$	$\int \gamma_5$	$\frac{\sqrt{4\pi}}{M}$	1	A
$\langle \Vert T_{10}(\vec{r},\vec{\sigma}) \Vert \rangle$	$\int \vec{\sigma}\cdot\vec{r}$	$\frac{\sqrt{4\pi}}{3}$	1	T-A-P
$\langle \Vert T_{10}(\vec{r},\vec{\sigma} \times \vec{p}) \Vert \rangle$	$\int \beta\vec{\alpha}$	$\frac{1}{M}\frac{4\pi}{\sqrt{3}}$	1	T
$\langle \Vert Y_{2}(\vec{r}) \Vert \rangle$	$\int R_{ij}$	$\left(\frac{8\pi}{15}\right)^{1/2}$	2	S - V
$\langle \Vert T_{21}(\vec{r},\vec{p}) \Vert \rangle$	$\int A_{ij}$	$\frac{8\pi}{3M}$	2	V
$\langle \Vert T_{22}(\vec{r},\vec{\sigma}) \Vert \rangle$	$\int T_{ij}$	$\frac{8\pi}{\sqrt{15}}$	2	T - A
$\langle \Vert T_{32}(\vec{r},\vec{\sigma}) \Vert \rangle$	$\int S_{ijk}$	$\left(\frac{96\pi^2}{15}\right)^{1/2}$	2	T - A
$\langle \Vert T_{21}(\vec{r},\vec{\sigma} \times \vec{p}) \Vert \rangle$	$\int A_{ij}^{\beta}$	$\frac{8\pi}{3M}$	2	T

Table 2. Parity and Forbiddenness of Operators

Operators	Parity	Forbiddenness
$\mathcal{Y}_{\lambda m}(\vec{r})$	$(-1)^{\lambda}$	λ
$T_{\lambda Lm}(\vec{r},\vec{\sigma})$	$(-1)^{L}$	L
$T_{\lambda Lm}(\vec{r},\vec{p})$	$(-1)^{L+1}$	$L+1$
$\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p}$	$(-1)^{\lambda+1}$	$\lambda+1$
$T_{\lambda Lm}(\vec{r},\vec{\sigma}x\vec{p})$	$(-1)^{L+1}$	$L+1$

So far, nothing has been said about the operators for electromagnetic transitions. It can be easily shown that the first three operators are also essentially responsible for electromagnetic radiation. The first one is responsible for the various electric moments. The second and the third operators are responsible for the multipole radiation of particles with spin.

It should be noted that the reduced matrix elements are not Hermitian. In many cases, the Hermitian conjugation is rather complicated since different arguments in the operator do not commute. In general, if $T_{\lambda m}$ denotes any one of the five operators, its Hermitian conjugate $T^{+}_{\lambda m}$ is given by

$$T^{+}_{\lambda m} = (-1)^{m} \left\{ (-1)^{\eta} T_{\lambda,-m} + \Gamma T'_{\lambda,-m} \right\} , \qquad (3\text{-}4)$$

where Γ is a constant and T'_{λ} is another operator of the same rank. For each of the five operators, the results for $(-1)^{\eta}$, Γ, and T'_{λ} are given in Table 3.

Now the relation between a reduced matrix element and its Hermitian conjugate may be determined in the following manner. First, from the Wigner-Eckart theorem, the following is obtained.

$$\langle f \mid T_{\lambda m} \mid i \rangle^{*} = \langle J_i \lambda M_i m \mid J_f M_f \rangle \langle f \parallel T_{\lambda} \parallel i \rangle^{*} . \qquad (3\text{-}5)$$

Also, with the help of Equation (3-4) for T^{+}_{λ} it follows that

Table 3. Tensor Operators under Hermitian Conjugation

$T_{\lambda m}$	$(-1)^{\eta}$	Γ	$T^{\dagger}_{\lambda,-m}$
$\mathcal{Y}_{\lambda m}(\vec{r})$	1	0	
$T_{\lambda Lm}(\vec{r},\vec{\sigma})$	$(-1)^{\lambda+L+1}$	0	
$T_{\lambda Lm}(\vec{r},\vec{p})$	$(-1)^{\lambda+L+1}$	$i(\frac{3}{4\pi}\cdot\frac{L}{2L-1})^{1/2}(2L+1)\delta_{\lambda,L-1}$	$\mathcal{Y}_{\lambda,-m}(\vec{r})$
$\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p}$	1	$-i[\frac{4\pi}{3}\lambda(2\lambda+1)]^{1/2}$	$T_{\lambda,\lambda-1,-m}(\vec{r},\vec{\sigma})$
$T_{\lambda Lm}(\vec{r},\vec{\sigma}x\vec{p})$	$(-1)^{\lambda+L+1}$	$\sqrt{6L}(2L+1)W(11\ L-1\ L,1\lambda)$	$T_{\lambda,L-1,-m}(\vec{r},\vec{\sigma})$

$$\langle f | T_{\lambda m} | i \rangle^* = \langle i | T^+_{\lambda m} | f \rangle ,$$

$$= (-1)^m \langle i | (-1)^\eta T_{\lambda,-m} + \Gamma T'_{\lambda,-m} | f \rangle ,$$

$$= (-1)^m \langle J_f \lambda M_f - m | J_i M_i \rangle \{ (-1)^\eta \langle i \| T_\lambda \| f \rangle + \Gamma \langle i \| T'_\lambda \| f \rangle \} ,$$

$$= (-1)^{J_i - J_f} \left(\frac{2J_i+1}{2J_f+1} \right)^{\frac{1}{2}} \langle J_i \lambda M_i m | J_f M_f \rangle \{ (-1)^\eta \langle i \| T_\lambda \| f \rangle + \Gamma \langle i \| T'_\lambda \| f \rangle . \quad (3\text{-}6)$$

Here, the symmetry properties of C-coefficients (Appendix I) have been used in order to obtain the last equation. Thus, from Equations (3-5) and (3-6) one has

$$\langle f \| T_\lambda \| i \rangle^* = (-1)^{J_i - J_f} \left(\frac{2J_i+1}{2J_f+1} \right)^{1/2} \{ (-1)^\eta \langle i \| T_\lambda \| f \rangle + \Gamma \langle i \| T'_\lambda \| f \rangle \} \quad (3\text{-}7)$$

In order to establish the results of Table 3 one simply starts from the definition of the spherical tensor. For the first two operators, it is quite obvious. As an example, the relation between $T_{\lambda Lm}(r,p)$ and $T^+_{\lambda Lm}(r,p)$ will be derived. From the definition (3-1)

$$T_{\lambda L m}(\vec{r},\vec{p}) = \sum_{m'} \langle 1\, L\, -m'\ m'+m | \lambda\, m \rangle\, \mathcal{Y}_{L,m'+m}(\vec{r})\ \mathcal{Y}_{1,-m'}(\vec{p}) . \quad (3\text{-}8)$$

Then

$$T^+_{\lambda L m}(\vec{r},\vec{p}) = \sum_{m'} \langle 1\, L\, -m'\ m'+m | \lambda\, m \rangle\, \mathcal{Y}^+_{1,-m'}(\vec{p})\ \mathcal{Y}^+_{L,m'+m}(\vec{r})$$

$$= (-1)^m \sum_{m'} \langle 1\, L\, -m'\ m'+m | \lambda\, m \rangle\, \mathcal{Y}_{1m'}(\vec{p})\ \mathcal{Y}_{L,-m'-m}(\vec{r}) . \quad (3\text{-}9)$$

Making use of the commutation relation (see note at the end of this chapter),

$$\left[Y_{l\mu}(\hat{r}), Y_{L\mu'}(\hat{r})\right] = i\left(\frac{3}{4\pi}\frac{L}{2L-1}\right)^{1/2}(2L+1)\langle l\,L\,\mu\mu' \,|\, L-1\;\mu+\mu'\rangle\, Y_{L-1,\mu+\mu'}(\hat{r}) \quad , \tag{3-10}$$

Equation (3-9) becomes

$$T^{\dagger}_{\lambda L m}(\vec{r},\vec{p}) = (-1)^{m}\sum_{m'}\langle l\,L\,-m'\,m'+m \,|\, \lambda\, m\rangle\, Y_{L,-m'-m}(\hat{r})\, Y_{l\,m'}(\hat{p})$$

$$+(-1)^{m} i\left(\frac{3}{4\pi}\frac{L}{2L-1}\right)^{1/2}(2L+1)\sum_{m'}\langle l\,L\,-m'\,m'+m \,|\, \lambda\, m\rangle\langle l\,L\,m'-m'-m \,|\, L-1\;-m\rangle\, Y_{L-1,-m}(\hat{r}) \quad . \tag{3-11}$$

Again, the symmetry properties of the C-coefficients are used. The first sum over m' yields $T_{\lambda L,-m}(r,p)$, by definition; and the second sum over m' yields $\delta_{\lambda,L-1}$. Therefore,

$$T^{\dagger}_{\lambda L m}(\vec{r},\vec{p}) = (-1)^{m}\left\{(-1)^{\lambda+L+1}\, T_{\lambda L,-m}(\vec{r},\vec{p}) + i\left(\frac{3}{4\pi}\frac{L}{2L-1}\right)^{1/2}(2L+1)\,\delta_{\lambda,L-1}\, Y_{L-1,-m}(\hat{r})\right\} \tag{3-12}$$

This equation is the desired result.

Note: The commutation rule (3-10) essentially comes from the gradient formula:

$$\nabla_{m} Y_{Lm'}\,\phi(r) = \left(\frac{L+1}{2L+1}\right)^{1/2}\langle L\;1\;m'\,m \,|\, L+1\;m'+m\rangle\, Y_{L+1,m'+m}\, D_{-}\,\phi(r)$$

$$-\left(\frac{L}{2L-1}\right)^{1/2}\langle L\;1\;m'\,m \,|\, L-1\;m'+m\rangle\, Y_{L-1,m'+m}\, D_{+}\,\phi(r) \quad , \tag{3-13}$$

where $\phi(r)$ is an arbitrary function of r and

$$D_{-} = \frac{d}{dr} - \frac{L}{r} \qquad\qquad D_{+} = \frac{d}{dr} + \frac{L+1}{r}$$

Now $$\mathcal{Y}_{1m}(\vec{p}) = \sqrt{\tfrac{3}{4\pi}}\, p_m = -i\sqrt{\tfrac{3}{4\pi}}\,\nabla_m \,,$$

$$\mathcal{Y}_{Lm'}(\vec{r}) = r^L Y_{Lm'} \,, \qquad (3\text{-}14)$$

So that

$$\begin{aligned}\left[\nabla_m, r^L Y_{Lm'}\right] &= \nabla_m (r^L Y_{Lm'}) - r^L Y_{Lm'} \nabla_m \\ &= \nabla_m r^L Y_{Lm'} + r^L Y_{Lm'} \nabla_m - r^L Y_{Lm'} \nabla_m \\ &= -\left(\tfrac{L}{2L-1}\right)^{1/2} (2L+1) \langle L\,1\; m'\,m \,|\, L-1\; m'+m\rangle\, r^{L-1} Y_{L-1,\,m'+m} \,, \qquad (3\text{-}15)\end{aligned}$$

or, finally,

$$\left[\mathcal{Y}_{1m}(\vec{p}), \mathcal{Y}_{Lm'}(\vec{r})\right] = i\left(\tfrac{3}{4\pi}\cdot\tfrac{L}{2L-1}\right)^{1/2} (2L+1) \langle 1\,L\; m\,m' \,|\, L-1\; m+m'\rangle \mathcal{Y}_{L-1,\,m+m'}(\vec{r}) \,. \qquad (3\text{-}10)$$

CHAPTER IV

NILSSON SINGLE PARTICLE INTRINSIC MATRIX ELEMENTS

For strongly deformed nuclei, the wave function

$$\Psi(JMK\Omega) = \left(\frac{2J+1}{16\pi^2}\right)^{1/2}\left\{D^J_{MK}\chi_\Omega + (-1)^{J-\frac{1}{2}}D^J_{M-K}\chi_{-\Omega}\right\} \tag{4-1}$$

in general gives a good representation of nuclear states. The characteristics of this wave function have been extensively discussed in Chapter II. The description of a particular nuclear level in terms of ψ is considered in Chapter V.

As discussed in Chapter III, the beta decay reduced nuclear matrix elements are proportional to matrix elements of the form

$$\langle f \mid T_{\lambda L m_s} \mid i \rangle \tag{4-2}$$

where $T_{\lambda L m_s}$ represents the spherical operators expressed in terms of a coordinate system fixed in space. In the evaluation of the matrix element (4-2) the integration over the variables of the rotational wave functions, D^J_{MK}, can be performed. Also, for situations in which χ_Ω represents the antisymmetrized product of two or more Nilsson single-particle wave functions, the integration over the intrinsic variables for the non-transforming particles can be carried out. Performing these two integrations makes it possible to express the matrix element (4-2) in terms of single-particle intrinsic matrix elements

$$\langle \chi_{\Omega_f} | T_{\lambda L m} | \chi_{\Omega_i} \rangle , \tag{4-3}$$

where $X_{\Omega i}$ and $X_{\Omega f}$ are respectively the initial and final Nilsson single-particle states of the transforming particle. $T_{\lambda L m}$ represents the spherical operators expressed in terms of a coordinate system fixed in the nuclear frame in which the Nilsson single-particle wave functions are described.

In this chapter the single-particle intrinsic matrix elements (4-3) are evaluated for the spherical operators, $T_{\lambda L m}$, defined in Chapter III. In Chapters VI and VII the expressions for the single-particle intrinsic matrix elements will be used in the derivation of the beta decay nuclear matrix elements for odd-mass and even-mass nuclei, respectively.

The Nilsson single-particle intrinsic wave function may be expressed as

$$\chi_{\Omega} = \sum_{\ell \Lambda} a_{\ell \Lambda} | N \ell \Lambda \Sigma \rangle , \tag{4-4}$$

where

$$| N \ell \Lambda \Sigma \rangle \sim e^{-\frac{1}{2} r^2} , F_1 \left(-n, \ell + \tfrac{3}{2}, r^2\right) Y_{\ell \Lambda} f_{s \Sigma} , \tag{4-5}$$

and

$$\Omega = \Lambda + \Sigma . \tag{4-6}$$

$, F_1 \left(-n, \ell + \frac{3}{2}, r^2\right)$ is the confluent hypergeometric function, $Y_{\ell \Lambda}$ the customary spherical harmonics, and $f_{s \Sigma}$ the spin wave function (with $s = 1/2$). Details of the wave function (4-4) can be found in Appendix II. The properties of the C-coefficient and the Racah coefficient which are used

throughout the chapter are discussed in Appendix I.

Finally, whenever the sum over i and f appears, it has the meaning of summing over $\ell_i, \Lambda_i, \Sigma_i$ and $\ell_f, \Lambda_f, \Sigma_f$, respectively. This "i" has nothing to do with the imaginary i ($i^2=-1$) which appears from time to time. The expressions such as $\langle Y_{\ell_f\Lambda_f} | Y_{\lambda m} | Y_{\ell_i\Lambda_i} \rangle$ have the meaning of an integral. Therefore,

$$\langle Y_{\ell_f\Lambda_f} | Y_{\lambda m} | Y_{\ell_i\Lambda_i} \rangle = \int Y^*_{\ell_f\Lambda_f} Y_{\lambda m} Y_{\ell_i\Lambda_i} d\omega. \tag{4-7}$$

Operator $\mathcal{Y}_{\lambda m}(\vec{r})$

From the definition of the solid spherical harmonics

$$\mathcal{Y}_{\lambda m}(\vec{r}) = r^\lambda Y_{\lambda m} ,$$

the matrix element $\langle X_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r}) | X_{\Omega_i} \rangle$ may be evaluated as follows:

$$\langle X_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r}) | X_{\Omega_i} \rangle = \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f} \langle N_f \ell_f \Lambda_f \Sigma_f | r^\lambda Y_{\lambda m} | N_i \ell_i \Lambda_i \Sigma_i \rangle$$

$$= \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f} \langle N_f \ell_f | r^\lambda | N_i \ell_i \rangle \langle Y_{\ell_i\Lambda_i} | Y_{\lambda m} | Y_{\ell_i\Lambda_i} \rangle \langle \Sigma_f | \Sigma_i \rangle$$

$$= \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f} \mathcal{F}_\lambda \left[\frac{(2\ell_i+1)(2\lambda+1)}{4\pi(2\ell_f+1)} \right]^{1/2} \langle \ell_i \lambda\, 0\, 0 | \ell_f\, 0 \rangle$$

$$\times \langle \ell_i \lambda\, \Lambda_i\, m | \ell_f \Lambda_f \rangle \delta_{\Sigma_i\Sigma_f} .$$

Thus

$$\langle X_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r}) | X_{\Omega_i} \rangle = (-1)^\lambda \left(\frac{2\lambda+1}{4\pi} \right)^{1/2} \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f} \mathcal{F}_\lambda \langle \ell_f \lambda\, 0\, 0 | \ell_i\, 0 \rangle$$

$$\times \langle \ell_i \lambda\, \Lambda_i\, m | \ell_f \Lambda_f \rangle \delta_{\Sigma_i\Sigma_f} , \tag{4-8}$$

where $$\mathcal{F}_\lambda = \langle N_f l_f | r^\lambda | N_i l_i \rangle$$

Details of the radial matrix element $\mathcal{F}_\lambda$ in this section and in the sections which follow may be found in Appendix II.

Operator $T_{\lambda L m}(\vec{r},\vec{\sigma})$

From the definition

$$T_{\lambda L m}(\vec{r},\vec{\sigma}) = \sum_{m'} \langle 1\, L\, -m'\, m'+m | \lambda m \rangle\, \mathcal{Y}_{L,m'+m}(\vec{r})\, \mathcal{Y}_{1,-m'}(\vec{\sigma})$$

one has for $\langle X_{\Omega_f} | T_{\lambda L m}(\vec{r},\vec{\sigma}) | X_{\Omega_i} \rangle$

$$\langle X_{\Omega_f} | T_{\lambda L m}(\vec{r},\vec{\sigma}) | X_{\Omega_i} \rangle = \sum_{i,f} a_{l_i\lambda_i} a_{l_f\lambda_f} \langle N_f l_f \Lambda_f \Sigma_f | T_{\lambda L m}(\vec{r},\vec{\sigma}) | N_i l_i \Lambda_i \Sigma_i \rangle$$

$$= \sum_{i,f} a_{l_i\Lambda_i} a_{l_f\Lambda_f} \langle N_f l_f | r^L | N_i l_i \rangle \sum_{m'} \langle 1\, L\, -m'\, m'+m | \lambda m \rangle$$

$$\times \langle Y_{l_f\Lambda_f} | Y_{L,m'+m} | Y_{l_i\Lambda_i} \rangle \langle \Sigma_f | \mathcal{Y}_{1,-m'}(\vec{\sigma}) | \Sigma_i \rangle . \qquad (4\text{-}9)$$

Making use of the relation[15]

$$\langle \Sigma_f | \mathcal{Y}_{1,-m'}(\vec{\sigma}) | \Sigma_i \rangle = \frac{3}{\sqrt{4\pi}} \langle \tfrac{1}{2}\, \Sigma_i\, -m' | \tfrac{1}{2} \Sigma_f \rangle ,$$

Equation (4-9) becomes

$$\sum_{i,f} a_{l_i\Lambda_i} a_{l_f\Lambda_f} \mathcal{F}_L \sum_{m'} \langle 1\, L\, -m'\, m'+m | \lambda m \rangle \left[\frac{(2l_i+1)(2L+1)}{4\pi(2l_f+1)} \right]^{\frac{1}{2}}$$

$$\times \frac{3}{\sqrt{4\pi}} \langle l_i\, L\, 0\, 0 | l_f\, 0 \rangle \langle l_i\, L\, \Lambda_i\, m'+m | l_f \Lambda_f \rangle \langle \tfrac{1}{2} | \Sigma_i\, -m' | \tfrac{1}{2} \Sigma_f \rangle$$

$$= (-1)^L \frac{3}{4\pi} (2L+1)^{\frac{1}{2}} \sum_{i,f} a_{l_i\Lambda_i} a_{l_f\Lambda_f} \mathcal{F}_L \langle l_f\, L\, 0\, 0 | l_i\, 0 \rangle$$

$$\times \langle L\, | \Sigma_f - \Sigma_i - m\, -\Sigma_f + \Sigma_i | \lambda - m \rangle \langle l_i\, L\, \Lambda_i\, -\Sigma_f + \Sigma_i + m | l_f \Lambda_f \rangle \langle \tfrac{1}{2} | \Sigma_i\, \Sigma_f - \Sigma_i | \tfrac{1}{2} \Sigma_f \rangle . \qquad (4\text{-}10)$$

Since the Σ's can only assume the values $\pm 1/2$ one has explicitly

$$\left\langle \tfrac{1}{2} \mid \Sigma_i \; \Sigma_f - \Sigma_i \mid \tfrac{1}{2} \; \Sigma_f \right\rangle = \frac{1}{\sqrt{3}} (-1)^{\Sigma_i - \frac{1}{2}} \delta_{\Sigma_i \Sigma_f} \,, \qquad (4\text{-}11a)$$

$$= \sqrt{\frac{2}{3}} \, \delta_{\Sigma_i \frac{1}{2}} \, \delta_{\Sigma_f, -\frac{1}{2}} \,, \qquad (4\text{-}11b)$$

$$= -\sqrt{\frac{2}{3}} \, \delta_{\Sigma_i, -\frac{1}{2}} \, \delta_{\Sigma_f \frac{1}{2}} \,. \qquad (4\text{-}11c)$$

Combining Equations (4-10) and (4-11), one obtains finally

$$\langle \chi_{\Omega_f} \mid T_{\lambda L m}(\vec{r}, \vec{\sigma}) \mid \chi_{\Omega_i} \rangle = (-1)^L \frac{3}{4\pi} \left(\frac{2L+1}{3} \right)^{1/2} \sum_{i,f} a_{\ell_i \Lambda_i} \, a_{\ell_f \Lambda_f} \, \mathcal{F}_L$$
$$\times \langle \ell_f \, L \; 0 \; 0 \mid \ell_i \; 0 \rangle \, A(\lambda L m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f) \,, \qquad (4\text{-}12)$$

where

$$A(\lambda L m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f) = \langle L \; 1 \; -m \; 0 \mid \lambda \; -m \rangle \langle \ell_i \; L \; \Lambda_i \; m \mid \ell_f \; \Lambda_f \rangle (-1)^{\Sigma_i - \frac{1}{2}} \delta_{\Sigma_i \Sigma_f}$$
$$+ \sqrt{2} \, \langle L \; 1 \; -1-m \; \; 1 \mid \lambda \; -m \rangle \langle \ell_i \; L \; \Lambda_i \; 1+m \mid \ell_f \; \Lambda_f \rangle \, \delta_{\Sigma_i \frac{1}{2}} \, \delta_{\Sigma_f, -\frac{1}{2}}$$
$$- \sqrt{2} \, \langle L \; 1 \; \; 1-m \; -1 \mid \lambda \; -m \rangle \langle \ell_i \; L \; \Lambda_i \; -1+m \mid \ell_f \; \Lambda_f \rangle \, \delta_{\Sigma_i, -\frac{1}{2}} \, \delta_{\Sigma_f \frac{1}{2}} \,. \qquad (4\text{-}13)$$

Operator $T_{\lambda L m}(\vec{r}, \vec{p})$

By definition, $T_{\lambda L m}(\vec{r}, \vec{p})$ may be written as

$$T_{\lambda L m}(\vec{r}, \vec{p}) = \sum_{m'} \langle 1 \; L \; -m' \; m'+m \mid \lambda \; m \rangle \, \mathcal{Y}_{L \, m'+m}(\vec{r}) \, \mathcal{Y}_{1, -m'}(\vec{p}) \,.$$

In spherical basis, $\mathcal{Y}_{1, -m'}(\vec{p})$ is the form

$$\mathcal{Y}_{1, -m'}(\vec{p}) = \left(\frac{3}{4\pi} \right)^{1/2} p_{-m'} = -i \left(\frac{3}{4\pi} \right)^{1/2} \nabla_{-m'} \,.$$

Thus

$$T_{\lambda L m}(\vec{r}, \vec{p}) = -i \left(\frac{3}{4\pi} \right)^{1/2} \sum_{m'} \langle 1 \; L \; -m' \; m'+m \mid \lambda \; m \rangle \, \mathcal{Y}_{L \, m'+m}(\vec{r}) \; \nabla_{-m'} \,. \qquad (4\text{-}14)$$

Later, the following gradient formula will be needed:

$$\nabla_{m'} Y_{\ell m}\Phi(r) = \left(\frac{\ell+1}{2\ell+3}\right)^{1/2}\langle \ell\ 1\ m\ m' | \ell+1\ m+m'\rangle Y_{\ell+1,m+m'} D_-(\ell)\Phi(r)$$

$$-\left(\frac{\ell}{2\ell-1}\right)^{1/2}\langle \ell\ 1\ m'\ m | \ell-1\ m+m'\rangle Y_{\ell-1,m+m'} D_+(\ell)\Phi(r)\ , \qquad (4\text{-}15)$$

where $D_-(\ell) = \frac{d}{dr} - \frac{\ell}{r}$ and $D_+(\ell) = \frac{d}{dr} + \frac{\ell+1}{r}$.

The matrix element for $T_{\lambda L m}(\vec{r},\vec{p})$ can be evaluated as follows:

$$\langle \chi_{\Omega_f} | T_{\lambda L m}(\vec{r},\vec{p}) | \chi_{\Omega_i}\rangle = \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f}\langle N_f \ell_f \Lambda_f \Sigma_f | T_{\lambda L m}(\vec{r},\vec{p}) | N_i \ell_i \Lambda_i \Sigma_i\rangle$$

$$= -i\left(\frac{3}{4\pi}\right)^{1/2}\sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f}\sum_{m'}\langle 1\ L\ -m'\ m'+m | \lambda m\rangle$$

$$\times\langle N_f \ell_f \Lambda_f \Sigma_f | \mathcal{Y}_{L,m'+m}(\vec{r})\nabla_{-m'} | N_i \ell_i \Lambda_i \Sigma_i\rangle\ . \qquad (4\text{-}16)$$

Letting $\nabla_{-m'}$ operate on the right side, one has (Equation (4-15))

$$\nabla_{-m'} | N_i \ell_i \Lambda_i \Sigma_i\rangle = \left(\frac{\ell_i+1}{2\ell_i+3}\right)^{1/2}\langle \ell_i\ 1\ \Lambda_i\ -m' | \ell_i+1\ \Lambda_i - m'\rangle Y_{\ell_i+1,\Lambda_i-m'} D_-(\ell_i) | N_i \ell_i\rangle$$

$$-\left(\frac{\ell_i}{2\ell_i-1}\right)^{1/2}\langle \ell_i\ 1\ \Lambda_i\ -m' | \ell_i-1\ \Lambda_i-m'\rangle Y_{\ell_i-1,\Lambda_i-m'} D_+(\ell_i) | N_i \ell_i\rangle\ .$$

After some simplification, the right side of Equation (4-16) reads

$$-i\frac{\sqrt{3}}{4\pi}\sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f}(-1)^{\ell_i+1}\delta_{\Sigma_i\Sigma_f}\sum_{m'}\langle 1\ L\ -m'\ m'+m | \lambda\ m\rangle$$

$$\times\Big\{(\ell_i+1)^{1/2}\mathcal{F}_L^-\langle \ell_i+1\ \ell_f\ 0\ 0 | L\ 0\rangle\langle \ell_i\ 1\ \Lambda_i\ -m' | \ell_i+1\ \Lambda_i-m'\rangle\langle \ell_i+1\ L\ \Lambda_i-m'\ m'+m | \ell_f\Lambda_f\rangle$$

$$-\ell_i^{1/2}\mathcal{F}_L^+\langle \ell_i-1\ \ell_f\ 0\ 0 | L\ 0\rangle\langle \ell_i\ 1\ \Lambda_i\ -m' | \ell_i-1\ \Lambda_i-m'\rangle\langle \ell_i-1\ L\ \Lambda_i-m'\ m'+m | \ell_f\Lambda_f\rangle\Big\}$$

$$= -i\frac{\sqrt{3}}{4\pi} \sum_{i,f} a_{\ell_i\Lambda_i}\, a_{\ell_f\Lambda_f}\, (-1)^{\ell_i+1}\, \delta_{\Sigma_i\Sigma_f} \sum_{\ell} (2\ell+1)^{1/2} \langle \ell_i\, \ell\, \Lambda_i\, m \,|\, \ell_f\, \Lambda_f\rangle$$

$$\times \sum_{m'} \langle 1\, L\, -m'\, m'+m \,|\, \lambda\, m\rangle \langle 1\, L\, -m'\, m'+m \,|\, \ell\, m\rangle$$

$$\times \Big\{ \left[(\ell_i+1)(2\ell_i+3)\right]^{1/2} \mathcal{F}_L^{-} \langle \ell_i+1\ \ell_f\ 0\ 0 \,|\, L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i+1\ \ell)$$

$$- \left[\ell_i(2\ell_i-1)\right]^{1/2} \mathcal{F}_L^{+} \langle \ell_i-1\ \ell_f\ 0\ 0 \,|\, L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i-1\ \ell) \Big\}. \quad (4\text{-}17)$$

The sum over m′ in Equation (4-17) yields $\delta_{\lambda\ell}$. One then sums over ℓ and obtains

$$\langle \chi_{\Omega_f} | T_{\lambda L m}(\vec{r},\vec{p}) | \chi_{\Omega_i} \rangle = -i\frac{\sqrt{3}}{4\pi} (2\lambda+1)^{1/2} \sum_{i,f} a_{\ell_i\Lambda_i}\, a_{\ell_f\Lambda_f}\, (-1)^{\ell_i+1}$$

$$\times \Big\{ \left[(\ell_i+1)(2\ell_i+3)\right]^{1/2} \mathcal{F}_L^{-} \langle \ell_i+1\ \ell_f\ 0\ 0 \,|\, L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i+1\ \lambda)$$

$$- \left[\ell_i(2\ell_i-1)\right]^{1/2} \mathcal{F}_L^{+} \langle \ell_i-1\ \ell_f\ 0\ 0 \,|\, L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i-1\ \lambda) \Big\}$$

$$\times \langle \ell_i\, \lambda\, \Lambda_i\, m \,|\, \ell_f\, \Lambda_f\rangle\, \delta_{\Sigma_i\Sigma_f}, \quad (4\text{-}18)$$

where $$\mathcal{F}_L^{\pm} = \langle N_f \ell_f | r^L D_{\pm} | N_i \ell_i \rangle .$$

Operator $\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p}$

From the definition of the scalar product, one may write

$$\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p} = \sum_{m'} (-1)^{m'} \mathcal{Y}_{\lambda m}(\vec{r})\, \sigma_{m'}\, p_{-m'},$$

or
$$\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p} = -i\left(\frac{4\pi}{3}\right)^{1/2} \sum_{m'} (-1)^{m'} \mathcal{Y}_{\lambda m}(\vec{r})\, \mathcal{Y}_{1\,m'}(\vec{\sigma})\, \nabla_{-m'} . \quad (4\text{-}19)$$

The matrix element for $\mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p}$ is then written

$$\langle \chi_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p} | \chi_{\Omega_i} \rangle = \sum_{i,f} a_{\ell_i\Lambda_i}\, a_{\ell_f\Lambda_f} \langle N_f \ell_f \Lambda_f \Sigma_f | \mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p} | N_i \ell_i \Lambda_i \Sigma_i \rangle . \quad (4\text{-}20)$$

The integration of Equation (4-20) may be carried out in a straightforward manner; using Equation (4-19) and the results obtained earlier, the right-hand side of Equation (4-20) then becomes

$$(-1)^{\lambda} i \sqrt{\frac{3}{4\pi}} (2\lambda+1)^{1/2} \sum_{i,f} a_{\ell_i \Lambda_i} a_{\ell_f \Lambda_f} \sum_{m'} \langle \tfrac{1}{2} \; | \; \Sigma_i \; m' \; | \; \tfrac{1}{2} \; \Sigma_f \rangle$$

$$\Big\{ \Big(\frac{\ell_i+1}{2\ell_i+1}\Big)^{1/2} \langle \ell_f \lambda \; 0 \; 0 \; | \; \ell_i+1 \; 0 \rangle \langle \ell_i+1 \; | \; \Lambda_i - m' \; m' \; | \; \ell_i \Lambda_i \rangle \langle \ell_i+1 \; \lambda \; \Lambda_i - m' \; m \; | \; \ell_f \Lambda_f \rangle \; \mathcal{F}_{\lambda}^{-}$$

$$- \Big(\frac{\ell_i}{2\ell_i+1}\Big)^{1/2} \langle \ell_f \lambda \; 0 \; 0 \; | \; \ell_i - 1 \; 0 \rangle \langle \ell_i - 1 \; | \; \Lambda_i - m' \; m' \; | \; \ell_i \Lambda_i \rangle \langle \ell_i - 1 \; \lambda \; \Lambda_i - m' \; m \; | \; \ell_f \Lambda_f \rangle$$

$$= (-1)^{\lambda} i \sqrt{\frac{3}{4\pi}} (2\lambda+1)^{1/2} \sum_{i,f} a_{\ell_i \Lambda_i} a_{\ell_f \Lambda_f} \langle \tfrac{1}{2} \; | \; \Sigma_i \; \Sigma_f - \Sigma_i \; | \; \tfrac{1}{2} \; \Sigma_f \rangle$$

$$\times \Big\{ \Big(\frac{\ell_i+1}{2\ell_i+1}\Big)^{1/2} \langle \ell_f \lambda \; 0 \; 0 \; | \; \ell_i+1 \; 0 \rangle \langle \ell_i+1 \; | \; \Lambda_i - \Sigma_f + \Sigma_i \; \Sigma_f - \Sigma_i \; | \; \ell_i \Lambda_i \rangle$$

$$\times \langle \ell_i+1 \; \lambda \; \Lambda_i - \Sigma_f + \Sigma_i \; m \; | \; \ell_f \Lambda_f \rangle \; \mathcal{F}_{\lambda}^{-}$$

$$- \Big(\frac{\ell_i}{2\ell_i+1}\Big)^{1/2} \langle \ell_f \lambda \; 0 \; 0 \; | \; \ell_i - 1 \; 0 \rangle \langle \ell_i - 1 \; | \; \Lambda_i - \Sigma_f + \Sigma_i \; \Sigma_f - \Sigma_i \; | \; \ell_i \Lambda_i \rangle$$

$$\times \langle \ell_i - 1 \; \lambda \; \Lambda_i - \Sigma_f + \Sigma_i \; m \; | \; \ell_f \Lambda_f \rangle \; \mathcal{F}_{\lambda}^{+} \Big\}, \qquad (4\text{-}21)$$

or finally:

$$\langle \chi_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r}) \, \vec{\sigma} \cdot \vec{p} \, | \, \chi_{\Sigma_i} \rangle = (-1)^{\lambda} i \Big(\frac{2\lambda+1}{4\pi}\Big)^{1/2} \sum_{i,f} a_{\ell_i \Lambda_i} a_{\ell_f \Lambda_f}$$

$$\times \Big\{ \Big(\frac{\ell_i+1}{2\ell_i+1}\Big)^{1/2} \langle \ell_f \lambda \; 0 \; 0 \; | \; \ell_i+1 \; 0 \rangle \, \mathcal{F}_{\lambda}^{-} \, B(\lambda \; \ell_i+1 \; m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f)$$

$$- \Big(\frac{\ell_i}{2\ell_i+1}\Big)^{1/2} \langle \ell_f \lambda \; 0 \; 0 \; | \; \ell_i - 1 \; 0 \rangle \, \mathcal{F}_{\lambda}^{+} \, B(\lambda \; \ell_i - 1 \; m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f) \Big\}, \qquad (4\text{-}22)$$

where

$$B(\lambda \; \ell_i \pm 1 \; m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f) = \langle \ell_i \pm 1 \; | \; \Lambda_i \; 0 \; | \; \ell_i \; \Lambda_i \rangle \langle \ell_i \pm 1 \; \lambda \; \Lambda_i \; m \; | \; \ell_f \Lambda_f \rangle (-1)^{\Sigma_i - \frac{1}{2}} \delta_{\Sigma_i \Sigma_f}$$

$$+ \sqrt{2} \, \langle \ell_i \pm 1 \; | \; \Lambda_i + 1 \; -1 \; | \; \ell_i \; \Lambda_i \rangle \langle \ell_i \pm 1 \; \lambda \; \Lambda_i + 1 \; m \; | \; \ell_f \Lambda_f \rangle \delta_{\Sigma_i \frac{1}{2}} \delta_{\Sigma_f, -\frac{1}{2}}$$

$$- \sqrt{2} \, \langle \ell_i \pm 1 \; | \; \Lambda_i - 1 \; 1 \; | \; \ell_i \; \Lambda_i \rangle \langle \ell_i \pm 1 \; \lambda \; \Lambda_i - 1 \; m \; | \; \ell_f \Lambda_f \rangle \delta_{\Sigma_i, -\frac{1}{2}} \delta_{\Sigma_f \frac{1}{2}} \, . \qquad (4\text{-}23)$$

The expression (4-21) may also be written in terms of Racah coefficients. One then has

$$\langle \chi_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p} \,|\, \chi_{\Omega_i}\rangle = (-1)^{\lambda+m} i\sqrt{\tfrac{3}{4\pi}}(2\lambda+1)^{1/2} \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f}$$

$$\times \sum_{\ell} (2\ell+1)^{1/2} \Big\{ \Big[\frac{(\ell_i+1)(2\ell_i+3)}{2\ell_i+1}\Big]^{1/2} \langle \ell_i+1\ \lambda\ 0\ 0 | \ell_f\ 0\rangle W(\ell_f\lambda\ell_i 1, \ell_i+1\ \ell)\mathcal{F}_\lambda^-$$

$$-\Big[\frac{\ell_i(2\ell_i-1)}{2\ell_i+1}\Big]^{1/2} \langle \ell_i-1\ \lambda\ 0\ 0 | \ell_f\ 0\rangle W(\ell_f\lambda\ell_i 1, \ell_i-1\ \ell)\mathcal{F}_\lambda^+ \Big\}$$

$$\times \langle \lambda\ 1\ -m\ \Sigma_f-\Sigma_i | \ell\ -m+\Sigma_f-\Sigma_i\rangle \langle \ell_f\ \ell\ \Lambda_f\ -m+\Sigma_f-\Sigma_i | \ell_i\Lambda_i\rangle \langle \tfrac{1}{2}\ 1\ \Sigma_i\ \Sigma_f-\Sigma_i | \tfrac{1}{2}\ \Sigma_f\rangle ,$$

or

$$\langle \chi_{\Omega_f} | \mathcal{Y}_{\lambda m}(\vec{r})\vec{\sigma}\cdot\vec{p} \,|\, \chi_{\Omega_i}\rangle = (-1)^{\lambda+m} i\Big(\frac{2\lambda+1}{4\pi}\Big)^{1/2} \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f}$$

$$\times \sum_{\ell} (2\ell+1)^{1/2} \Big\{ \Big[\frac{(\ell_i+1)(2\ell_i+3)}{2\ell_i+1}\Big]^{1/2} \langle \ell_i+1\ \lambda\ 0\ 0 | \ell_f\ 0\rangle W(\ell_f\lambda\ell_i 1, \ell_i+1\ \ell)\mathcal{F}_\lambda^-$$

$$-\Big[\frac{\ell_i(2\ell_i-1)}{2\ell_i+1}\Big]^{1/2} \langle \ell_i-1\ \lambda\ 0\ 0 | \ell_f\ 0\rangle W(\ell_f\lambda\ell_i 1, \ell_i-1\ \ell)\mathcal{F}_\lambda^+ \Big\} C(\lambda\ell m, \Lambda_i\Sigma_i\Lambda_f\Sigma_f), \quad (4\text{-}24)$$

where

$$C(\lambda\ell m, \Lambda_i\Sigma_i\Lambda_f\Sigma_f) = \langle \lambda\ 1\ -m\ 0 | \ell\ -m\rangle \langle \ell_f\ \ell\ \Lambda_f\ -m | \ell_i\Lambda_i\rangle (-1)^{\Sigma_i-\frac{1}{2}} \delta_{\Sigma_i\Sigma_f}$$

$$+\sqrt{2}\langle \lambda\ 1\ -m\ -1 | \ell\ -m-1\rangle \langle \ell_f\ \ell\ \Lambda_f\ -m-1 | \ell_i\Lambda_i\rangle \delta_{\Sigma_i\frac{1}{2}}\delta_{\Sigma_f,-\frac{1}{2}}$$

$$-\sqrt{2}\langle \lambda\ 1\ -m\ 1 | \ell\ -m+1\rangle \langle \ell_f\ \ell\ \Lambda_f\ -m+1 | \ell_i\Lambda_i\rangle \delta_{\Sigma_i,-\frac{1}{2}}\delta_{\Sigma_f\frac{1}{2}} . \quad (4\text{-}25)$$

In this form the sum over ℓ is rather cumbersome, but the radial-dependent part can be factored out. It is particularly simple when $\lambda=0$ which corresponds to $\int\gamma_5$. This matrix element, $\int\gamma_5$, is frequently encountered. In this case the sum over ℓ can be carried out, hence

$$\langle \chi_{\Omega_f} | \vec{\sigma}\cdot\vec{p} | \chi_{\Omega_i} \rangle = i \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f} \left\{ \left(\frac{\ell_i+1}{2\ell_i+1}\right)^{1/2} \mathcal{G}_0^- \delta_{\ell_i+1,\ell_f} - \left(\frac{\ell_i}{2\ell_i+1}\right)^{1/2} \mathcal{G}_0^+ \delta_{\ell_i-1,\ell_f} \right\} C(0\,|\,0, \Lambda_i\Sigma_i\Lambda_f\Sigma_f), \tag{4-26}$$

where

$$C(0\,|\,0, \Lambda_i\Sigma_i\Lambda_f\Sigma_f) = \langle \ell_f\, |\, \Lambda_f\, 0 | \ell_i\, \Lambda_i \rangle (-1)^{\Sigma_i - \frac{1}{2}} \delta_{\Sigma_i\Sigma_f} + \sqrt{2} \langle \ell_f\, |\, \Lambda_f - 1 \,|\, \ell_i\, \Lambda_i \rangle \delta_{\Sigma_i \frac{1}{2}} \delta_{\Sigma_f, -\frac{1}{2}} - \sqrt{2} \langle \ell_f\, |\, \Lambda_f\, 1 \,|\, \ell_i\, \Lambda_i \rangle \delta_{\Sigma_i, -\frac{1}{2}} \delta_{\Sigma_f \frac{1}{2}} . \tag{4-27}$$

Operator $T_{\lambda Lm}(\vec{r}, \vec{\sigma}\times\vec{p})$

One writes as usual

$$T_{\lambda Lm}(\vec{r}, \vec{\sigma}\times\vec{p}) = \sum_{m'} \langle 1\, L\, -m'\, m'+m \,|\, \lambda m \rangle \mathcal{Y}_{L,m'+m}(\vec{r}) \mathcal{Y}_{1,-m'}(\vec{\sigma}\times\vec{p}) .$$

Now $$\mathcal{Y}_{1,-m'}(\vec{\sigma}\times\vec{p}) = \left(\frac{3}{4\pi}\right)^{1/2} (\vec{\sigma}\times\vec{p})_{-m'} = i \left(\frac{8\pi}{3}\right)^{1/2} T_{1,1,-m'}(\vec{\sigma}, \vec{p}) .$$

Then again, by definition

$$T_{11,-m'}(\vec{\sigma}, \vec{p}) = \sum_{m''} \langle 1\,1\, -m''\, m''-m' \,|\, 1\, -m' \rangle \mathcal{Y}_{1,m''-m'}(\vec{\sigma}) \mathcal{Y}_{1,-m''}(\vec{p}),$$

and $T_{\lambda Lm}(\vec{r}, \vec{\sigma}\times\vec{p})$ takes the final form

$$T_{\lambda Lm}(\vec{r}, \vec{\sigma}\times\vec{p}) = \sqrt{2} \sum_{m', m''} \langle 1\, L\, -m'\, m'+m \,|\, \lambda m \rangle \langle 1\,1\, -m''\, m''-m' \,|\, 1\, -m' \rangle \times \mathcal{Y}_{L,m'+m}(\vec{r}) \mathcal{Y}_{1,m''-m'}(\vec{\sigma}) \nabla_{-m''} . \tag{4-28}$$

The matrix element for $T_{\lambda Lm}(\vec{r}, \vec{\sigma}\times\vec{p})$ is evaluated as follows:

$$\langle \chi_{\Omega_f} | T_{\lambda Lm}(\vec{r}, \vec{\sigma}\times\vec{p}) | \chi_{\Omega_i} \rangle = \sqrt{2} \sum_{i,f} a_{\ell_i\Lambda_i} a_{\ell_f\Lambda_f} \sum_{m', m''} \langle 1\, L\, -m'\, m'+m \,|\, \lambda\, m \rangle \times \langle 1\,1\, -m''\, m''-m' | 1\, -m' \rangle \langle N_f \ell_f \Lambda_f \Sigma_f \,|\, \mathcal{Y}_{L,m'+m}(\vec{r}) \mathcal{Y}_{1,m''-m'}(\vec{\sigma}) \nabla_{-m''} | N_i \ell_i \Lambda_i \Sigma_i \rangle . \tag{4-29}$$

After integrations, the right-hand side of Equation (4-29) reads

$$\frac{3\sqrt{5}}{4\pi}\sum_{i,f} a_{\ell_i\lambda_i} a_{\ell_f\lambda_f} \sum_{m',m''} \langle 1\, L\, -m'\; m'+m | \lambda\, m\rangle \langle 1\, 1\, -m''\; m''-m' | 1\, -m'\rangle$$
$$\times (-1)^{\ell_i+1} \langle \tfrac{1}{2}\, | \Sigma_i\; m''-m' | \tfrac{1}{2}\, \Sigma_f\rangle$$
$$\times \Big\{ (\ell_i+1)^{\frac{1}{2}} \langle \ell_i\, 1\, \lambda_i\, -m'' | \ell_i+1\; \lambda_i - m''\rangle \langle \ell_i+1\; L\; \lambda_i - m''\; m'+m | \ell_f \Lambda_f\rangle \langle \ell_i+1\; \ell_f\, 0\, 0 | \ell_i\, 0\rangle \mathcal{F}_L^-$$
$$- \ell_i^{1/2} \langle \ell_i\, 1\, \lambda_i - m'' | \ell_i - 1\; \lambda_i - m''\rangle \langle \ell_i - 1\; L\; \lambda_i - m''\; m'+m | \ell_f \Lambda_f\rangle \langle \ell_i - 1\; \ell_f\, 0\, 0 | \ell_i\, 0\rangle \mathcal{F}_L^+ \Big\}$$

$$= \frac{3\sqrt{2}}{4\pi}\sum_{i,f} a_{\ell_i\lambda_i} a_{\ell_f\lambda_f} \sum_{m',m''} \langle 1\, L\, -m'\; m'+m | \lambda\, m\rangle \langle 1\, 1\, -m''\; m''-m | 1\, -m'\rangle \langle \tfrac{1}{2}\, | \Sigma_i\, m''-m' | \tfrac{1}{2}\, \Sigma_f\rangle (-1)^{\ell_i+1}$$
$$\times \sum_{\ell} (2\ell+1)^{1/2} \langle 1\, L\, -m''\; m'+m | \ell\, -m''+m'+m\rangle \langle \ell_i\, \ell\, \lambda_i\, -m''+m'+m | \ell_f \Lambda_f\rangle$$
$$\times \Big\{ \big[(\ell_i+1)(2\ell_i+3)\big]^{\frac{1}{2}} \langle \ell_i+1\; \ell_f\, 0\, 0 | L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i+1\; \ell)\, \mathcal{F}_L^-$$
$$- \big[\ell_i(2\ell_i-1)\big]^{1/2} \langle \ell_i - 1\; \ell_f\, 0\, 0 | L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i - 1\; \ell)\, \mathcal{F}_L^+ \Big\}$$

$$= \frac{3\sqrt{2}}{4\pi}\sum_{i,f} a_{\ell_i\lambda_i} a_{\ell_f\lambda_f} (-1)^{\ell_i+1} \sum_{\ell m''} (2\ell+1)^{1/2} \langle \tfrac{1}{2}\, | \Sigma_i\; \Sigma_f - \Sigma_i | \tfrac{1}{2}\, \Sigma_f\rangle \langle \ell_i\, \ell\, \Lambda_i\, -\Sigma_f+\Sigma_i+m | \ell_f \Lambda_f\rangle$$
$$\times \langle 1\, L\; \Sigma_f - \Sigma_i\; -m''\; -\Sigma_f+\Sigma_i+m''+m | \lambda\, m\rangle \langle 1\, 1\; -m''\; \Sigma_f - \Sigma_i | 1\; -m''+\Sigma_f-\Sigma_i\rangle$$
$$\times \langle 1\, L\; -m''\; -\Sigma_f+\Sigma_i+m''+m | \ell\; -\Sigma_f+\Sigma_i+m\rangle$$
$$\times \Big\{ \big[(\ell_i+1)(2\ell_i+3)\big]^{\frac{1}{2}} \langle \ell_i+1\; \ell_f\, 0\, 0 | L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i+1\; \ell)\, \mathcal{F}_L^-$$
$$- \big[\ell_i(2\ell_i-1)\big]^{\frac{1}{2}} \langle \ell_i - 1\; \ell_f\, 0\, 0 | L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i - 1\; \ell)\, \mathcal{F}_L^+ \Big\}$$

$$= \frac{3\sqrt{6}}{4\pi}\sum_{i,f} a_{\ell_i\lambda_i} a_{\ell_f\lambda_f} (-1)^{\ell_i} \sum_{\ell} (2\ell+1)\, W(1\, 1\, \lambda\, L, 1\, \ell)$$
$$\times \Big\{ \big[(\ell_i+1)(2\ell_i+3)\big]^{\frac{1}{2}} \langle \ell_i+1\; \ell_f\, 0\, 0 | L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i+1\; \ell)\, \mathcal{F}_L^-$$
$$- \big[\ell_i(2\ell_i-1)\big]^{\frac{1}{2}} \langle \ell_i - 1\; \ell_f\, 0\, 0 | L\, 0\rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i - 1\; \ell)\, \mathcal{F}_L^+ \Big\}$$

$$\times \langle l\, \ell\, \Sigma_f - \Sigma_i \; -\Sigma_f + \Sigma_i + m \,|\, \lambda m \rangle \langle \ell_i\, \ell\, \Lambda_i \; -\Sigma_f + \Sigma_i + m \,|\, \ell_f \Lambda_f \rangle \langle \tfrac{1}{2}\, 1\, \Sigma_i \; \Sigma_f - \Sigma_i \,|\, \tfrac{1}{2}\, \Sigma_f \rangle .$$

The final expression for the matrix element for $T_{\lambda L m}(\vec{r}, \vec{\sigma} \times \vec{p})$ is

$$\langle \chi_{\Omega_f} | T_{\lambda L m}(\vec{r}, \vec{\sigma} \times \vec{p}) | \chi_{\Omega_i} \rangle = \frac{3\sqrt{2}}{4\pi} \sum_{i,f} a_{\ell_i \Lambda_i}\, a_{\ell_f \Lambda_f} (-1)^{\ell_i} \sum_{\ell} (2\ell+1) W(11\lambda L, 1\ell)$$

$$\times \{ [(\ell_i+1)(2\ell_i+3)]^{\frac{1}{2}} \langle \ell_i+1\; \ell_f\; 00 | L0 \rangle\, W(\ell_i\, 1\, \ell_f\, L, \ell_i+1\, \ell)\, F_L^-$$

$$- [\ell_i(2\ell_i-1)]^{\frac{1}{2}} \langle \ell_i-1\; \ell_f\; 00 | L0 \rangle W(\ell_i\, 1\, \ell_f\, L, \ell_i-1\, \ell)\, F_L^+ \}\, D(\lambda \ell m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f) . \qquad (4\text{-}30)$$

where

$$D(\lambda \ell m, \Lambda_i \Sigma_i \Lambda_f \Sigma_f) = \langle 1\, \ell' 0\, m \,|\, \lambda m \rangle \langle \ell_i\, \ell\, \Lambda_i\; m \,|\, \ell_f \Lambda_f \rangle (-1)^{\Sigma_i - \frac{1}{2}} \delta_{\Sigma_i \Sigma_f}$$

$$+ \sqrt{2} \langle 1\, \ell\, -1\; 1+m \,|\, \lambda m \rangle \langle \ell_i\, \ell\, \Lambda_i\; 1+m \,|\, \ell_f \Lambda_f \rangle\, \delta_{\Sigma_i \frac{1}{2}}\, \delta_{\Sigma_f, -\frac{1}{2}}$$

$$- \sqrt{2} \langle 1\, \ell\; 1\; -1+m \,|\, \lambda m \rangle \langle \ell_i\, \ell\, \Lambda_i\; -1+m \,|\, \ell_f \Lambda_f \rangle\, \delta_{\Sigma_i, -\frac{1}{2}}\, \delta_{\Sigma_f \frac{1}{2}} .$$

CHAPTER V

NILSSON MODEL DESCRIPTION OF NUCLEAR LEVELS

For the unified model of the nucleus, the appropriately symmetrized total nuclear wave function for strongly deformed nuclei has the form

$$\Psi(JMK\Omega) = \left(\frac{2J+1}{16\pi^2}\right)^{1/2} \phi_{vib} \left\{ D^J_{MK} \chi_\Omega + (-1)^{J-\sum j_i} D^J_{M-K} \chi_{-\Omega} \right\} . \qquad (5\text{-}1)$$

It has been assumed that there is rotational symmetry around the intrinsic nuclear axis and reflection symmetry through a plane perpendicular to this axis. This wave function is characterized by the quantum numbers J,M,K and Ω . In this chapter the determination of the appropriate quantum numbers and corresponding wave function for a particular nuclear level will be discussed.

Characteristics of Total Nuclear Wave Function

First, the form of the total nuclear wave function and the definitions of the appropriate quantum numbers will be briefly discussed. Φ_{vib} describes the vibrations of the nucleus around its equilibrium shape. However, throughout his investigation, Nilsson[17] assumes that the nucleus is always in the vibrational ground state. The same assumption is made in the present investigation. Therefore Φ_{vib} may be set equal to 1.

D^J_{MK} presents the collective rotational motion of the nucleus as a whole. Since the nucleus has an axis of symmetry, the rotational motion

is particularly simple and is characterized by the quantum numbers J, M, and K. The meaning of J, M, and K is illustrated in Figure 1 (p. 19). The total angular momentum $\vec{J}$ is the sum of $\vec{R}$, the angular momentum generated by the collective motion of the nucleus and $\vec{j}$ the angular momentum due to the intrinsic motion of the nucleons. The quantum numbers J, M, and K represent the total angular momentum, its projection along the space fixed axis (Oz) and its projection along the nuclear symmetry axis (Oz'), respectively.

X_Ω represents the intrinsic motion, which can be expressed in terms of the motion of individual particles in a stationary deformed field. In the coupling scheme appropriate for large deformations the individual nucleons move independently in the deformed nuclear field. In this case, the intrinsic motion is characterized by the constants of motion Ω_i, the component of the angular momentum $\vec{j}_i$ of each nucleon along the nuclear symmetry axis, Oz'. $\vec{j}$, the total intrinsic angular momentum, is not a constant of the motion, though its component along Oz' is a good quantum number and is denoted by Ω, where $\Omega = \sum_i \Omega_i$. As a consequence, the total intrinsic wave function, X_Ω, is simply the antisymmetrized product of individual-particle wave functions X_{Ω_i} when the nucleons in question are identical. $X_{\Omega i}$ is the Nilsson single-particle wave function appropriate for the ith nucleon. Apart from accidental degeneracies, the states are doubly degenerate (corresponding to $\pm \Omega_i$).

In Equation (5-1), the phase $(-1)^{J-\Sigma j_i}$ is thought of as a matrix when j_i, the angular momentum of the ith particle, is not a constant of the motion. The normalization factor comes from the normalization of the rotational wave functions, D^J_{MK}.

In the vibrational ground state, $\vec{R}$ is perpendicular to Oz′ and therefore $\Omega=K$. Since $\Omega(=\Omega_i)$ is determined by the intrinsic state of the nucleus, K is entirely a property of the intrinsic motion in this situation. This simplification ($\Omega=K$) exists throughout the present invesitgation.

The Nilsson model is expected to provide a valid description of nuclear properties in regions of the atomic chart where there are sufficient particles outside of closed shells to cause the nuclear shape to deviate appreciably from spherical symmetry. These regions are roughly $A\sim 25$, $150 < A < 190$, and $A > 200$. Therefore, it should be possible to describe the ground state and excited levels of nuclei in this region with wave functions having the form of Equation (5-1) Mottelson and Nilsson[22] have predicted the appropriate Nilsson model description for the experimentally observed energy levels of odd-A nuclei in these regions. Gallagher and Solovier[23] have made a similar analysis for even-A nuclei. Generally speaking, each energy level of a particular nucleus is characterized as being a certain intrinsic state with corresponding quantum number Ω. Associated with each intrinsic state there is a rotational band. The energy levels of a rotational band are all represented by the same intrinsic state and therefore have the same Ω and K (since $K=\Omega$). However, the levels in a given band will have different J values.

In the following sections the determination of the appropriate Nilsson model description of a particular nuclear level is discussed. The determination of the intrinsic state of a level will be discussed first for ground states and then for excited states. A discussion of the rotational quantum numbers (J, M and K) associated with a particular intrinsic state will then be presented.

Description of Intrinsic Ground States

The appropriate intrinsic state and the corresponding value of Ω for a particular nuclear level is determined in the following manner. The component of angular momentum along the symmetry axis, Ω_i, is a constant of the motion for each particle and the total Ω equals $\Sigma\Omega_i$. Each Nilsson single-particle energy state is degenerate corresponding to $\pm\,\Omega_i$. Since the nucleus contains two groups "a" and "b" of equivalent particles - neutrons and protons - it is assumed the particles of each group fill pairwise in the levels independently of the other group. In this situation each group of particles will be independently characterized by a value of Ω (either Ω_a or Ω_b) obtained by summing over the particles in the group.

Now for the ground state of a particular nucleus the following assumptions are made by Nilsson[17]. If the number of particles in group "a" is even, the Ω_i values for the individual particles cancel by pairs so that $\Omega_a = 0$. If the number of particles in group "a" is odd, Ω_a equals the Ω_i of the last unpaired particle. The Nilsson single-particle intrinsic state along with its value of Ω_i for the last unpaired particle are determined as follows. Nilsson has plotted the single-particle intrinsic energy levels as a function of the deformation for both neutrons and protons. The proper intrinsic level is found by placing two particles to a level on the appropriate energy level diagram until the level for the last particle is reached. In the above situation the total Ω for the nucleus becomes

$$\Omega = | \Omega_a \pm \Omega_b | \qquad (5\text{-}2)$$

Odd-Mass Nuclei

The proper description of intrinsic states for odd-mass nuclei has been considered by Mottelson and Nilsson[22].

For the ground state of an odd-mass nucleus Mottelson and Nilsson suggest an intrinsic state X_Ω that is simply the Nilsson single-particle state appropriate for the odd particle. As described above, the proper single-particle state for the odd particle is found by adding particles pairwise on the energy level diagram. In this situation Ω equals the Ω_i of the single-particle state of the odd particle. Throughout the present investigation, single-particle Nilsson states are used to describe the intrinsic states of odd-mass nuclei for both ground and excited states.

Even-Mass Nuclei

The determination of the proper intrinsic state representation for even-mass nuclei has been considered by Gallagher[18]. Gallagher suggests that for most purposes the ground states of even-mass nuclei can be characterized by two-particle intrinsic states. In this situation, the $X_{\pm\Omega}$ of Equation (5-1) are products of two Nilsson single-particle states:

$$\chi_\Omega = \chi_{\Omega_1} \chi_{\pm\Omega_2} , \qquad (5\text{-}3a)$$

$$\chi_{-\Omega} = \chi_{-\Omega_1} \chi_{\mp\Omega_2} . \qquad (5\text{-}3b)$$

The sign of Ω_2 is determined by the coupling of the particles. The subscripts 1 and 2 indicate that this general form applies to neutron-proton, proton-proton, and proton-neutron systems. Throughout the present

investigation two-particle intrinsic states as proposed by Gallagher are used to describe both the ground and excited states of even-mass nuclei.

Odd-Odd Nuclei. For the ground state of an odd-odd nucleus the two-particle intrinsic state suggested by Gallagher is formed by coupling the appropriate Nilsson single-particle states for the odd proton and the odd neutron. The Nilsson single-particle states for each particle are determined from the corresponding energy level diagrams in the same manner as for the odd particle in an odd-mass nucleus. The intrinsic wave function will have the form given by Equation (5-3a).

$$\chi_{\Omega} = \chi_{\Omega_p} \chi_{\pm\Omega_n} , \tag{5-4}$$

where the sign of Ω_n is determined from the coupling which is applicable to the particular case.

The corresponding value of Ω for the ground state of an odd-odd nucleus will have the form given by Equation (5-2)

$$\Omega = |\Omega_p \pm \Omega_n| , \tag{5-5}$$

where Ω_p and Ω_n characterize the Nilsson single-particle states of the odd proton and the odd neutron, respectively. Note that for the proton-neutron system, two intrinsic states are possible, corresponding to the parallel or antiparallel orientation of the total angular momenta of the two particles along the symmetry axis. It is necessary to determine which of the two intrinsic states given by Equations (5-4) and (5-5) actually represent the ground state of a particular odd-odd nucleus. Gallagher and Moszkowski[19] have shown that empirical evidence indicates

that the ground state is the one in which the intrinsic spins (denoted by the quantum number Σ in Equation (2-49d)) of the particles couple parallel.

Even-Even Nuclei. The ground state of an even-even nucleus consists of pairs of nucleons coupled to spin zero. In this situation $\Omega=0$. If a two-particle intrinsic state is used to describe the nucleus only one pair of particles can be represented in the wave function. The pair chosen for representation is generally the pair involved in the particular calculation under consideration. For a proton pair in the ground state of an even-even nucleus the intrinsic wave function is

$$\chi_{\Omega} = \chi^{(1)}_{\Omega_p} \chi^{(2)}_{-\Omega_p} \quad . \tag{5-6}$$

Since identical particles are represented in the wave function, the superscripts are necessary to distinguish between the two particles.

For situations where the wave function is required to be fully symmetrized, the intrinsic wave function for the proton pair becomes

$$\chi_{\Omega} = \frac{1}{\sqrt{2}} \left[\chi^{(1)}_{\Omega_p} \chi^{(2)}_{-\Omega_p} - \chi^{(1)}_{-\Omega_p} \chi^{(2)}_{\Omega_p} \right] \quad . \tag{5-7}$$

For a neutron pair in the ground state of an even-even nucleus, the intrinsic wave function will have the same form as for a proton pair with Ω_n replacing Ω_p.

As an example of a two-particle intrinsic wave function chosen to represent the ground state of an even-even nucleus consider the following. Suppose it is desired to calculate the matrix elements for a β^- transition

from the intrinsic ground state of an odd-odd nucleus to the intrinsic ground state of an even-even nucleus. In this situation a neutron is transformed into a proton. In order to make the calculation, the transforming particle must obviously be represented in the initial and final nuclear wave functions. Therefore the proton pair which includes the transforming nucleon as a member must be represented in the final wave function.

Description of Intrinsic Excited States

Odd-Mass Nuclei

Now, consider the possibility of describing the excited energy levels of strongly deformed nuclei in terms of the Nilsson model. Mottelson and Nilsson[22] have suggested that the excited energy levels of odd-mass nuclei can be interpreted as excited single-particle intrinsic states. As usual, there is a rotational band associated with each intrinsic exceted state. Mottelson and Nilsson describe the intrinsic excited states as the single-particle Nilsson state of the odd particles. This is essentially the same procedure that is used for the intrinsic ground states of odd-mass nuclei except for the following difference. For the intrinsic ground states it is generally assumed that the odd particle occupies the first available Nilsson state. However, the intrinsic excited levels are generally due to the odd particle occupying a Nilsson state higher up on the appropriate energy level diagram.

Even-Mass Nuclei

Gallagher and Soloviev[23] have discussed the Nilsson model description of the excited energy levels of even-mass nuclei. They

suggest that the excited levels of many strongly deformed even-mass nuclei can be interpreted as two-particle intrinsic excitations. For odd-odd nuclei the intrinsic excited states are interpreted as being proton-neutron states formed by coupling the Nilsson single-particle states of the odd proton and the odd neutron. This is essentially the same as the situation that exists for the intrinsic ground states of odd-odd nuclei. Therefore, the intrinsic wave functions, X_{Ω}, and its corresponding value of Ω will still be given by Equations (5-4) and (5-5), respectively. However, for the intrinsic ground state it is generally assumed that the proton and neutron occupy the lowest energy available Nilsson states and couple with their intrinsic spins parallel. Intrinsic excited states are formed by changing the coupling of the two particles or by placing one or both particles in higher energy Nilsson states.

Gallagher and Soloviev have described the intrinsic excited states of even-even nuclei as proton-proton or neutron-neutron excitations. In this situation, all of the nucleons do not couple by pairs to produce $\Omega = 0$ as they do for the intrinsic ground state of an even-even nucleus. For a proton-proton excitation the last two protons do not pair off but go into different available Nilsson states. As usual, the use of a two-particle intrinsic wave function means that the Nilsson states of only two particles can be represented in the wave function. For a proton-proton excitation, it would appear that the two unpaired protons must be represented. In this situation the two particle intrinsic wave function becomes

$$\chi_{\Omega} = \chi^{(1)}_{\Omega'_p}\, \chi^{(2)}_{\pm\Omega_p} \quad , \text{ or} \tag{5-8}$$

$$\chi_{\Omega} = \frac{1}{\sqrt{2}}\left[\chi^{(1)}_{\Omega'_p}\, \chi^{(2)}_{\pm\Omega_p} - \chi^{(1)}_{\pm\Omega_p}\, \chi^{(2)}_{\Omega'_p} \right] . \tag{5-9}$$

Equation (5-9) is applicable if the wave function must be fully symmetrized. Ω is no longer zero as in the ground state but is,

$$\Omega = \left| \Omega'_p \pm \Omega_p \right| . \tag{5-10}$$

For a neutron-neutron intrinsic excited state in an even-even nucleus the situation is similar to the proton-proton excitation with Ω_n and $\Omega_{n'}$ replacing Ω_p and $\Omega_{p'}$ in Equations (5-8), (5-9) and (5-10).

For the calculation of beta decay matrix elements for transitions to intrinsic excited states of even-even nuclei the following situation exists. The two-particle proton-proton and neutron-neutron wave functions discussed above are applicable for a β^- transition to a proton-proton excited state of a β^+ transition to a neutron-neutron excited state. However, they are not applicable for a β^- transition to a neutron-neutron excited state of a β^+ transition to a proton-proton excited state. This is due to the fact that in order to calculate the matrix elements the transforming particle must be represented in the final nuclear wave function.

Description of Rotational Motion

For the unified model of the nucleus there is a rotational band associated with each intrinsic state. Therefore, the states in a rotational

band are characterized by the same intrinsic wave function and the same value of Ω . The existence of the nuclear symmetry axis implies that K, the component of the total angular momentum along the symmetry axis, is a constant for each rotational band. The condition that the nucleus is in the vibrational ground state prevails throughout the present investigation. Therefore, $K=\Omega$. Hence, a determination of the intrinsic state of a particular nuclear level establishes the value of Ω and K for all levels in the associated rotational band. The symmetry of the nucleus with respect to reflection in a plane perpendicular to the symmetry axis means that the rotational motion always has even parity. Therefore the parity of all the states in a rotational band is the same and is equal to the parity of the intrinsic state.

The states in a rotational band are characterized by the same value of Ω and K, but they have different values of J. For an odd-mass nucleus K is a positive, half odd-integer and for the states of the rotational band J may take on the values

$$J=K,\ K+1,\ K+2,\ \ldots \qquad (5\text{-}11)$$

For the ground state of an even-even nucleus $K=0$ and symmetrization of the wave function limits the rotational band to

$$J=0,\ 2,\ 4,\ 6,\ \ldots \qquad (5\text{-}12a)$$

For odd-odd nuclei and for intrinsic excited states of even-even nuclei with $K \neq 0$, K is a positive integer and the rotational sequence is

$$J=K,\ K+1,\ K+2\ \ldots \qquad (5\text{-}12b)$$

CHAPTER VI

REDUCED MATRIX ELEMENTS FOR ODD-MASS NUCLEI IN BETA DECAY

For both even-A and odd-A nuclei the total nuclear wave function of the unified model has the general form given in Chapter V, Equation (5-1). In the present chapter the derivation of the beta decay reduced matrix elements for odd-A nuclei is made, using the wave function (5-1) in which ϕ_{vib} is set equal to 1 and X_Ω is the single-particle intrinsic wave function of the form

$$X_\Omega = \sum_{\ell\Lambda} a_{\ell\Lambda} |N\ell\Lambda\Sigma\rangle \quad \text{with} \quad \Omega = \Lambda+\Sigma \,, \tag{6-1a}$$

and

$$(-1)^{J-\frac{1}{2}} X_{-\Omega} = (-1)^{J+\ell-\frac{1}{2}} \sum_{\ell\Lambda} a_{\ell\Lambda} |N\ell-\Lambda-\Sigma\rangle \,. \tag{6-1b}$$

Since X_Ω is taken as a single-particle wave function representing the odd transforming nucleon, the expressions derived in this chapter are restricted to β^- and β^+ decays of odd-N and odd-Z nuclei, respectively. In the $\beta^-(\beta^+)$ decay of odd-Z (odd-N) nuclei it is not the last odd nucleon but one of the pair which transforms. Moreover, for odd-A nuclei K is a half odd-integer, in order to satisfy the condition

$$K = \Omega = \sum_i \Omega_i \tag{6-2}$$

it is necessary that an odd number of Ω_i be taken since Ω_i is always a half odd-integer. Hence, a three-particle intrinsic wave function ought

to be appropriate. This situation is not considered in the present chapter, however.

Since the intrinsic wave function is described in nuclear coordinates, in order to evaluate the matrix element for the operator $T_{\lambda Lm}$ the following transformation ought to be made. Let $T_{\lambda Lm_s}(s)$ be the operator described in the laboratory frame and $T_{\lambda Lm}(n)$ be the same operator but related to the nuclear frame, then $T_{\lambda Lm}(n)$ transforms according to (Equation (I-41), Appendix I)

$$T_{\lambda Lm_s}(s) = \sum_m D^{\lambda}_{m_s m}\, T_{\lambda Lm}(n) . \tag{6-3}$$

The transition matrix element $\langle f \mid T_{\lambda Lm_s} \mid i \rangle$ between the initial state $\mid i\rangle = \mid J_i M_i K_i \rangle$ and the final state $\mid f\rangle = \mid J_f M_f K_f \rangle$ will then be evaluated as follows.

$$\langle f \mid T_{\lambda Lm_s} \mid i \rangle = \langle J_f M_f K_f \mid T_{\lambda Lm_s} \mid J_i K_i M_i \rangle$$

$$= \frac{[(2J_i+1)(2J_f+1)]^{1/2}}{16\pi^2} \sum_m \langle D^{J_f}_{M_f K_f} \chi_{\Omega_f} + (-1)^{J_f - j_f} D^{J_f}_{M_f -K_f} \chi_{-\Omega_f} \mid D^{\lambda}_{m_s m}\, T_{\lambda Lm} \mid$$

$$D^{J_i}_{M_i K_i} \chi_{\Omega_i} + (-1)^{J_i - j_i} D^{J_i}_{M_i -K_i} \chi_{-\Omega_i} \rangle$$

$$= \frac{[(2J_i+1)(2J_f+1)]^{1/2}}{16\pi^2} \sum_m \Big\{ \langle D^{J_f}_{M_f K_f} \mid D^{\lambda}_{m_s m} \mid D^{J_i}_{M_i K_i} \rangle \langle \chi_{\Omega_f} \mid T_{\lambda Lm} \mid \chi_{\Omega_i} \rangle$$

$$+ (-1)^{J_i + J_f - j_i - j_f} \langle D^{J_f}_{M_f -K_f} \mid D^{\lambda}_{m_s m} \mid D^{J_i}_{M_i -K_i} \rangle \langle \chi_{-\Omega_f} \mid T_{\lambda Lm} \mid \chi_{-\Omega_i} \rangle$$

$$+(-1)^{J_f-j_f}\langle D^{J_f}_{M_f-K_f}\mid D^{\lambda}_{m_s m}\mid D^{J_i}_{M_i K_i}\rangle\langle \chi_{-\Omega_f}\mid T_{\lambda L m}\mid \chi_{\Omega_i}\rangle$$

$$+(-1)^{J_i-j_i}\langle D^{J_f}_{M_f K_f}\mid D^{\lambda}_{m_s m}\mid D^{J_i}_{M_i-K_i}\rangle\langle \chi_{\Omega_f}\mid T_{\lambda L m}\mid \chi_{-\Omega_i}\rangle\Big\}. \qquad (6\text{-}4)$$

By making use of the relations

$$\langle D^{J_f}_{M_f K_f}\mid D^{\lambda}_{m_s m}\mid D^{J_i}_{M_i K_i}\rangle = \frac{8\pi^2}{2J_f+1}\langle J_i\lambda M_i m_s\mid J_f M_f\rangle\langle J_i\lambda K_i m\mid J_f K_f\rangle ,$$

and

$$\langle f\mid T_{\lambda L m_s}\mid i\rangle = \langle J_i\lambda M_i m_s\mid J_f M_f\rangle\langle f\|T_{\lambda L}\|i\rangle ,$$

Equation (6-4) becomes

$$\begin{aligned}\langle f\|T_{\lambda L}\|i\rangle = \tfrac{1}{2}\left(\frac{2J_i+1}{2J_f+1}\right)^{1/2}\Big\{&\langle J_i\lambda K_i k\mid J_f K_f\rangle\langle \chi_{\Omega_f}\mid T_{\lambda L k}\mid \chi_{\Omega_i}\rangle\\ &+(-1)^{J_i+J_f-j_i-j_f}\langle J_i\lambda -K_i -k\mid J_f -K_f\rangle\langle \chi_{-\Omega_f}\mid T_{\lambda L,-k}\mid \chi_{-\Omega_i}\rangle\\ &+(-1)^{J_f-j_f}\langle J_i\lambda K_i q\mid J_f -K_f\rangle\langle \chi_{-\Omega_f}\mid T_{\lambda L q}\mid \chi_{\Omega_i}\rangle\\ &+(-1)^{J_i-j_i}\langle J_i\lambda -K_i -q\mid J_f K_f\rangle\langle \chi_{\Omega_f}\mid T_{\lambda L,-q}\mid \chi_{-\Omega_i}\rangle\Big\}.\end{aligned} \qquad (6\text{-}5)$$

where

$$k = K_f - K_i , \qquad (6\text{-}6a)$$

and

$$q = -K_f - K_i . \qquad (6\text{-}6b)$$

For every operator, the relation

$$\langle \chi_{-\Omega_f}\mid T_{\lambda L,-k}\mid \chi_{-\Omega_i}\rangle = (-1)^{l_i+\lambda-l_f}\langle \chi_{\Omega_f}\mid T_{\lambda L k}\mid \chi_{\Omega_i}\rangle$$

holds, together with the fact that J is a half odd-integer for odd-mass nuclei the general expression is obtained

$$\langle f \| T_{\lambda L} \| i \rangle = \left(\frac{2J_i+1}{2J_f+1} \right)^{1/2} \left\{ \langle J_i \lambda K_i k \mid J_f K_f \rangle \langle \chi_{\Omega_f} \mid T_{\lambda L k} \mid \chi_{\Omega_i} \rangle \right.$$
$$\left. + (-1)^{J_f - j_f} \langle J_i \lambda K_i q \mid J_f - K_f \rangle \langle \chi_{-\Omega_f} \mid T_{\lambda L q} \mid \chi_{\Omega_i} \rangle \right\} . \quad (6\text{-}7)$$

In the derivation of Equation (6-7), the factors $(-1)^{2J_i-1}$ and $(-1)^{2(J_i-J_f)}$ are encountered and set equal to one. This is of course true for a half odd-integral value of J. If the single-particle intrinsic wave function is used for even-mass nuclei, the factor $(-1)^{2J_i-1}$ will equal -1 since for even-mass nuclei, J is always an integer. Hence, Equation (6-7) reduces to

$$\langle f \| T_{\lambda L} \| i \rangle = (-1)^{J_f - j_f} \left(\frac{2J_i+1}{2J_f+1} \right)^{1/2} \langle J_i \lambda K_i q \mid J_f - K_f \rangle \langle \chi_{-\Omega_f} \mid T_{\lambda L q} \mid \chi_{\Omega_i} \rangle \ . \quad (6\text{-}8)$$

As was discussed at the beginning of the chapter, the use of the single-particle intrinsic wave function for even-mass nuclei appears pointless if not impossible since one should construct the intrinsic wave function X_Ω in such a way that $\Omega = K$. This is impossible for even-mass nuclei since K is always an integer while Ω always has a half odd-integral value for a single particle.

The final expressions for the reduced matrix elements for various operators can be obtained by substituting the intrinsic matrix elements in Equation (6-7) by the explicit expressions given in Chapter IV, with k and q replacing m. The change in sign of Ω_f corresponds to the change in signs of both Λ_f and Σ_f in the explicit expressions.

CHAPTER VII

REDUCED MATRIX ELEMENTS FOR EVEN MASS NUCLEI IN BETA DECAY

The total nuclear wave function for even-A nuclei has the general form given in Chapter V, Equation (5-1). In the present derivation of the beta decay reduced matrix elements, X_Ω is assumed to be a two-particle intrinsic wave function for even-A nuclei. As discussed in Chapter V, there are certain beta decays for which the two-particle intrinsic wave function does not sufficiently describe the situation.

The expressions derived in the present chapter are restricted to β^- and β^+ decays from odd-odd to even-even nuclei. For this situation the initial intrinsic wave function is a proton-neutron state and the final intrinsic wave function is either a proton-proton or neutron-neutron state. The form of X_Ω for these various possibilities is discussed at length in Chapter V and will be summarized here.

For the initial proton-neutron intrinsic state the following possibilities exist:

$$\chi_\Omega = \chi^{(1)}_{\Omega_p} \chi^{(2)}_{\Omega_n} \quad \text{with} \quad \Omega = \Omega_p + \Omega_n \,. \tag{7-1a}$$

$$\chi_\Omega = \chi^{(1)}_{\Omega_p} \chi^{(2)}_{-\Omega_n} \quad \text{with} \quad \Omega = \Omega_p - \Omega_n \,. \tag{7-1b}$$

$$\chi_\Omega = \chi^{(1)}_{-\Omega_p} \chi^{(2)}_{\Omega_n} \quad \text{with} \quad \Omega = -\Omega_p + \Omega_n \,. \tag{7-1c}$$

For Equation (7-1a) the proton and neutron have their angular momentum

components coupled parallel. For Equations (7-1b) and (7-1c) the angular momentum components are coupled anti-parallel. Since $K=\Omega$ is always positive, Equation (7-1b) is used for $\Omega_p>\Omega_n$ while Equation (7-1c) is used for $\Omega_n>\Omega_p$. The superscripts (1) and (2) denote particles (1) and (2) and are superfluous for the proton-neutron state since the particles are not identical.

For a final proton-proton intrinsic state the following possibilities exist:

$$\chi_\Omega = \frac{1}{\sqrt{2}}\left[\chi^{(1)}_{\Omega'_p}\chi^{(2)}_{\Omega_p} - \chi^{(1)}_{\Omega_p}\chi^{(2)}_{\Omega'_p}\right] \quad \text{with} \quad \Omega = \Omega'_p + \Omega_p . \tag{7-2a}$$

$$\chi_\Omega = \frac{1}{\sqrt{2}}\left[\chi^{(1)}_{\Omega'_p}\chi^{(2)}_{-\Omega_p} - \chi^{(1)}_{-\Omega_p}\chi^{(2)}_{\Omega'_p}\right] \quad \text{with} \quad \Omega = \Omega'_p - \Omega_p . \tag{7-2b}$$

For equations (7-2a) and (7-2b) the two protons have their angular momentum components coupled parallel and anti-parallel, respectively. Since $K=\Omega$ is always positive, for Equation (7-2b) $\Omega'_p>\Omega_p$.

For a final neutron-neutron intrinsic state the equations are identical to (7-2a) and (7-2b) with Ω'_n replacing Ω'_p and Ω_n replacing Ω_p.

Since the particles are identical for the proton-proton and neutron-neutron states, the wave functions (7-2a) and (7-2b) have been symmetrized and the superscripts (1) and (2) are necessary.

For a β^- decay for an odd-odd nucleus to an even-even nucleus, the initial intrinsic state is given by (7-1a), (7-1b) and (7-1c) and the final intrinsic state by (7-2a) or (7-2b). Therefore, there are six possible combinations of initial and final intrinsic states. These six cases may be divided into two categories based on the angular momentum

couplings of the two nucleons before and after the transition.

A transition is called a transition between states of the same relative coupling if the relative signs connecting the Ω's of the two particles are the same in the initial and final states. That is, the two particles are coupled parallel (or anti-parallel) in both the initial and final state. The transition is said to be between states of different relative couplings if the relative signs connecting the Ω's of the two particles are different in the initial and final states. That is, the two particles are coupled parallel in the initial state and anti-parallel in the final state or vice-versa.

Now for a β^- decay from an odd-odd nucleus to an even-even nucleus, the following six combinations of initial and final intrinsic states exist:

1) (7-1a)→(7-2a) or $\Omega_{pi}^{+}\ \Omega_{ni} \rightarrow \Omega_{pf}'^{+}\ \Omega_{pf}$.

2) (7-1a)→(7-2b) or $\Omega_{pi}^{+}\ \Omega_{ni} \rightarrow \Omega_{pf}'^{-}\ \Omega_{pf}$.

3) (7-1b)→(7-2a) or $\Omega_{pi}^{-}\ \Omega_{ni} \rightarrow \Omega_{pf}'^{+}\ \Omega_{pf}$.

4) (7-1b)→(7-2b) or $\Omega_{pi}^{-}\ \Omega_{ni} \rightarrow \Omega_{pf}'^{-}\ \Omega_{pf}$.

5) (7-1c)→(7-2a) or $\Omega_{ni}^{-}\ \Omega_{pi} \rightarrow \Omega_{pf}'^{+}\ \Omega_{pf}$.

6) (7-1c)→(7-2b) or $\Omega_{ni}^{-}\ \Omega_{pi} \rightarrow \Omega_{pf}'^{-}\ \Omega_{pf}$.

1, 4 and 6 represent transitions between states of the same relative coupling and 2, 3 and 5 represent transitions between states of different relative couplings.

Now equations may be derived that express the reduced matrix elements for a β^- decay from an odd-odd to an even-even nucleus in

terms of the Nilsson single-particle matrix elements of Chapter IV. Generally, the derivation proceeds in a manner similar to that used for odd-mass nuclei in Chapter VI.

The total nuclear wave function is given both initially and finally by Equation (5-1) where it is now understood that X_Ω represents the appropriate two-particle intrinsic wave function as expressed in Equations (7-1a,b,c) and (7-2a,b). Since the superscript (2) has been associated with the neutron in the initial state, for β^- decay the transforming particle is denoted as particle (2) and the non-transforming particle is particle (1). Therefore, for β^- decay the operator can always be assumed to operate on the wave function with the superscript (2). Hence, the scalar product of two single-particle intrinsic wave functions with the superscript (1) will appear in the calculations. These overlap integrals have the form:

$$\langle X^{(1)}_{\Omega_{p_f}} | X^{(1)}_{\Omega_{p_i}} \rangle \quad \text{or} \quad \langle X^{(1)}_{-\Omega_{p_f}} | X^{(1)}_{\Omega_{p_i}} \rangle \ .$$

Integrals such as $\langle X^{(1)}_{-\Omega p_f} | X^{(1)}_{\Omega p_i} \rangle$ are identically equal to zero due to the orthogonality of the single-particle wave functions. Integrals such as $\langle X^{(1)}_{\Omega p_f} | X^{(1)}_{\Omega p_i} \rangle$ are equal to zero if $X^{(1)}_{\Omega p_f} \neq X^{(1)}_{\Omega p_i}$ or one if $X^{(1)}_{\Omega p_f} = X^{(1)}_{\Omega p_i}$. This condition is expressed by replacing $\langle X^{(1)}_{\Omega p_f} | X^{(1)}_{\Omega p_i} \rangle$ with the symbol, $\delta_{\Omega p_f \Omega p_i}$.

Equations are presented for each of the six combinations of initial and final intrinsic states listed above. The general expressions for the reduced matrix elements in terms of single-particle intrinsic matrix elements for a β^- transition from an odd-odd to an even-even nucleus are presented below.

As previously discussed the equations are valid for a transition between an initial proton-neutron intrinsic state and a final proton-proton intrinsic state. Both initially and finally the states may be either the intrinsic ground state or an intrinsic excited state.

Furthermore, in addition to the general expressions, the expressions for the special case: $X_{\Omega p_f} = X_{\Omega p'_f}$ are presented. This special case applies to β^- transitions to the intrinsic ground state of the even-even nucleus and is encountered frequently.

General Case

Transitions between States of the Same Relative Coupling

$$\Omega_{p_i} + \Omega_{n_i} \rightarrow \Omega'_{p_f} + \Omega_{p_f}$$

$$\langle f \| T_{\lambda L} \| i \rangle = \left(\tfrac{1}{2} \cdot \frac{2J_i+1}{2J_f+1}\right)^{1/2} \Big\{ \langle J_i \lambda K_i k | J_f K_f \rangle \langle X_{\Omega_{p_f}} | T_{\lambda L k} | X_{\Omega_{n_i}} \rangle \delta_{\Omega_{p_i} \Omega'_{p_f}}$$

$$- \langle J_i \lambda K_i k | J_f K_f \rangle \langle X_{\Omega'_{p_f}} | T_{\lambda L k} | X_{\Omega_{n_i}} \rangle \delta_{\Omega_{p_i} \Omega_{p_f}} \Big\} . \qquad (7\text{-}3)$$

$$\Omega_{p_i} - \Omega_{n_i} \rightarrow \Omega'_{p_f} - \Omega_{p_f}$$

$$\langle f \| T_{\lambda L} \| i \rangle = \left(\tfrac{1}{2} \cdot \frac{2J_i+1}{2J_f+1}\right)^{1/2} \Big\{ \langle J_i \lambda K_i k | J_f K_f \rangle \langle X_{-\Omega_{p_f}} | T_{\lambda L k} | X_{\Omega_{n_i}} \rangle \delta_{\Omega_{p_i} \Omega'_{p_f}}$$

$$+ (-1)^{J_f - j_f - j_{f'}} \langle J_i \lambda K_i q | J_f - K_f \rangle \langle X_{-\Omega'_{p_f}} | T_{\lambda L q} | X_{\Omega_{n_i}} \rangle \delta_{\Omega_{p_i} \Omega_{p_f}} \Big\} \qquad (7\text{-}4)$$

$$-\Omega_{p_i}+\Omega_{n_i}\rightarrow \Omega'_{p_f}-\Omega_{p_f}$$

$$\langle f\|T_{\lambda L}\|i\rangle = -\left(\tfrac{1}{2}\cdot\frac{2J_i+1}{2J_f+1}\right)^{1/2}\Big\{\langle J_i\lambda K_i k|J_f K_f\rangle\langle \chi_{-\Omega'_{p_f}}|T_{\lambda Lk}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega_{p_f}}$$

$$+(-1)^{J_f-j_f-j'_f}\langle J_i\lambda K_i q|J_f-K_f\rangle\langle \chi_{\Omega_{p_f}}|T_{\lambda Lq}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega'_{p_f}}\Big\}. \qquad (7\text{-}5)$$

Transitions Between States of Different Relative Couplings

$$\Omega_{p_i}+\Omega_{n_i}\rightarrow \Omega'_{p_f}-\Omega_{p_f}$$

$$\langle f\|T_{\lambda L}\|i\rangle = \left(\tfrac{1}{2}\cdot\frac{2J_i+1}{2J_f+1}\right)^{1/2}\Big\{\langle J_i\lambda K_i k|J_f K_f\rangle\langle \chi_{-\Omega_{p_f}}|T_{\lambda Lk}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega'_{p_f}}$$

$$+(-1)^{J_f-j_f-j'_f}\langle J_i\lambda K_i q|J_f-K_f\rangle\langle \chi_{-\Omega'_{p_f}}|T_{\lambda Lq}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega_{p_f}}\Big\}. \qquad (7\text{-}6)$$

$$\Omega_{p_i}-\Omega_{n_i}\rightarrow \Omega'_{p_f}+\Omega_{p_f}$$

$$\langle f\|T_{\lambda L}\|i\rangle = \left(\tfrac{1}{2}\cdot\frac{2J_i+1}{2J_f+1}\right)^{1/2}\Big\{\langle J_i\lambda K_i k|J_f K_f\rangle\langle \chi_{-\Omega_{p_f}}|T_{\lambda Lk}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega'_{p_f}}$$

$$-\langle J_i\lambda K_i k|J_f K_f\rangle\langle \chi_{-\Omega'_{p_f}}|T_{\lambda Lk}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega_{p_f}}\Big\}. \qquad (7\text{-}7)$$

$$-\Omega_{p_i}+\Omega_{n_i}\rightarrow \Omega'_{p_f}+\Omega_{p_f}$$

$$\langle f\|T_{\lambda L}\|i\rangle = (-1)^{J_f-j_f-j'_f+1}\left(\tfrac{1}{2}\cdot\frac{2J_i+1}{2J_f+1}\right)^{1/2}\Big\{\langle J_i\lambda K_i q|J_f-K_f\rangle\langle \chi_{-\Omega_{p_f}}|T_{\lambda Lq}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega'_{p_f}}$$

$$-\langle J_i\lambda K_i q|J_f-K_f\rangle\langle \chi_{-\Omega'_{p_f}}|T_{\lambda Lq}|\chi_{\Omega_{n_i}}\rangle\delta_{\Omega_{p_i}\Omega_{p_f}}\Big\}. \qquad (7\text{-}8)$$

Special Case: $X_{\Omega p_f} = X_{\Omega p'_f}$

In this case, the proton-proton pair in the final state pair off in the same Nilsson single-particle level with their spin components coupled anti-parallel. This situation exists whenever the final state is the intrinsic ground state of the even-even nucleus. The majority of beta decays for which useful experimental results are available fall into this category. The final intrinsic wave function becomes:

$$\chi_{\Omega} = \frac{1}{\sqrt{2}}\left[\chi^{(1)}_{\Omega_p}\chi^{(2)}_{-\Omega_p} - \chi^{(1)}_{-\Omega_p}\chi^{(2)}_{\Omega_p}\right] \tag{7-9}$$

with $K_f = \Omega_f = \Omega_{p_f} - \Omega_{p_f} = 0$

For this special situation the following cases exist:

Initial State: $\Omega_{p_i} - \Omega_{n_i}$

$$\langle f\|T_{\lambda L}\|i\rangle = \left(\frac{2J_i+1}{2J_f+1}\right)^{\frac{1}{2}}\langle J_i \lambda K_i k | J_f 0\rangle\langle \chi_{\Omega\rho_f}|T_{\lambda L k}|\chi_{\Omega n_i}\rangle \delta_{\Omega\rho_i,\Omega\rho_f} \tag{7-10}$$

Initial State: $-\Omega_{p_i} + \Omega_{n_i}$

$$\langle f\|T_{\lambda L}\|i\rangle = \left(\frac{2J_i+1}{2J_f+1}\right)^{1/2}\langle J_i \lambda K_i k | J_f 0\rangle\langle \chi_{\Omega\rho_f}|T_{\lambda L k}|\chi_{\Omega n_i}\rangle \delta_{\Omega\rho_i,\Omega\rho_f} \tag{7-11}$$

Initial State: $\Omega_{p_i} + \Omega_{n_i}$

$$\langle f\|T_{\lambda L}\|i\rangle = \left(\frac{2J_i+1}{2J_f+1}\right)^{\frac{1}{2}}\langle J_i \lambda K_i k | J_f 0\rangle\langle \chi_{\Omega\rho_f}|T_{\lambda L k}|\chi_{\Omega n_i}\rangle \delta_{\Omega\rho_i,\Omega\rho_f} \tag{7-12}$$

The expressions for the reduced matrix elements presented above are specifically for a β^- transition from an odd-odd nucleus to an even-even nucleus. However, they are made applicable for a β^+ transition from an odd-odd nucleus to an even-even nucleus by the following changes.

$$\Omega_{n_i} \rightarrow \Omega_{p_i} \;,\; \Omega_{p_i} \rightarrow \Omega_{n_i} \;,\; \Omega_{p_f} \rightarrow \Omega_{n_f} \;,\; \Omega'_{p_f} \rightarrow \Omega'_{n_f}$$

$$j_{p_f} \rightarrow j_{n_f} \quad , \quad j'_{p_f} \rightarrow j'_{n_f}$$

These changes simply amount to replacing the subscript "n" with the subscript "p" and vice-versa throughout the equations. It is necessary to make the changes in both the matrix element equations and the identifying equations that state the Ω couplings for the various cases.

When modified for a β^+ transition the equations are valid for a transition between an initial proton-neutron intrinsic state and a final neutron-neutron intrinsic state. The initial and final states may be either the intrinsic ground state or an intrinsic excited state.

The equations presented in the present chapter are for either a β^- or a β^+ decay of an odd-odd nucleus. The beta decay of an even-even nucleus is rare in nature, and, therefore, is not considered. However, the derivation of the reduced matrix elements for the beta decay of an even-even nucleus can be made in the same manner as for the odd-odd nucleus.

CHAPTER VIII

SELECTION RULES

For very strong deformation the exact solution $X_\Omega = \Sigma a_{\ell\Lambda} |N\ell\Lambda\Sigma\rangle$ of the Nilsson potential may be substituted by the asymptotic solution $[Nn_z\Lambda\Sigma]$[17] in which the particle motion is separated into oscillations along the z-axis and those in the plane perpendicular to this axis. Due to this separation the new constants of the motion are the principal quantum number, N, the number of nodes along the z-axis, n_z, and the number of nodes in the x-y plane, Λ. In this limit, the expressions for the reduced matrix elements reduce to only one term. The selection rules for these asymptotic quantum numbers covering up to the second-forbidden beta decay were given by Alaga[24], using the single-particle intrinsic wave function.

In the present dissertation, since the exact solution of the Nilsson potential is used to evaluate the nuclear matrix elements, the selection rules for the corresponding quantum numbers are a little more complicated. However, they may be easily seen from the expressions for the matrix elements. Since the parity of the nuclear states is conserved, the selection rule for the parity, π, is the parity of the operator itself, given in Chapter III. The selection rule for the principal quantum number, N, results from the property of the orthogonal polynomials and is discussed in Appendix II. The selection rules for all the other quantum numbers, J, K, ℓ, Λ and Σ, come either from the triangle condition,

Equation (1-5a), or from the constraint among the projection quantum numbers in the C-coefficient, Equation (1-5b), or from the property of the projection of an angular momentum being not larger than the angular momentum itself.

Aside from the selection rules discussed above, the transition matrix elements also depend on the structure of the intrinsic wave function. For odd-A nuclei, the selection rules are unique since one-particle intrinsic wave functions are used, as can be seen from Equation (6-7). For even-A nuclei, two-particle wave functions are used. Therefore, the matrix elements also depend on the relative coupling between the initial and final states of the angular momenta of the two nucleons. Since this angular momentum coupling affects only the projection quantum number, Ω $(=\Lambda+\Sigma)$, as was shown in Chapter VIII, the selection rules for π, J, K, N, and ℓ are given separately in Table 4. For odd-A nuclei, the selection rules for Λ and Σ are given in Table 5. For even-A (odd N-odd Z) nuclei, the transition between states of the same or different relative couplings leads to many possibilities, each corresponds to a set of selection rules for Λ and Σ. Not all these cases, but two illustrative examples, are considered here. For the transition between states of the same relative coupling, the selection rules for Λ and Σ corresponding to Equations (7-4) or (7-10) are given in Table 6, and for the transition between states of different relative couplings, the selection rules for Λ and Σ corresponding to Equations (7-6), or (7-12) are given in Table 7. These two modes of transition will be applied later to the isotopes of Re^{186}_{75} and Tm^{170}, respectively. Since no confusion arises the subscripts n and p are suppressed, and Ω_{p_i} is replaced by Ω_i' in Tables 6 and 7.

Table 4. Selection Rules for π, J, K, N and ℓ .

Matrix Element	$\Delta\pi$	$\Delta J=\lvert J_f-J_i\rvert$	$k=K_f-K_i$ $q=K_f-K_i$	$\Delta N=\lvert N_f-N_i\rvert$	$\Delta\ell=\lvert\ell_f-\ell_i\rvert$
$\langle\lVert Y_\lambda(\vec{r})\rVert\rangle$	$(-1)^\lambda$			$\Delta N \leqslant \lambda$	$\Delta\ell \leqslant \lambda$ $\ell_i+\ell_f+\lambda$ even
$\langle\lVert T_{\lambda L}(\vec{r},\vec{\sigma})\rVert\rangle$	$(-1)^L$			$\Delta N \leqslant L$	$\Delta\ell \leqslant L$ $\ell_i+\ell_f+L$ even
$\langle\lVert T_{\lambda L}(\vec{r},\vec{p})\rVert\rangle$	$(-1)^{L+1}$	$\Delta J \leqslant \lambda$ $(J_i+J_f>\lambda)$	$\lvert k\rvert \leqslant \lambda$ $\lvert q\rvert \leqslant \lambda$	$\Delta N \leqslant L-1$ $\lvert N_f-N_i+1\rvert \leqslant L$ $\Delta N \leqslant L+1$	$\Delta\ell \leqslant L-1$ $\lvert\ell_f-\ell_i+1\rvert \leqslant L$ $\Delta\ell \leqslant L+1$ $\ell_i+\ell_f+L$ odd
$\langle\lVert Y_\lambda(\vec{r})\vec{\sigma}\cdot\vec{p}\rVert\rangle$	$(-1)^{\lambda+1}$			$\Delta N \leqslant \lambda-1$ $\lvert N_f-N_i+1\rvert \leqslant \lambda$ $\Delta N \leqslant \lambda+1$	$\Delta \leqslant \lambda-1$ $\lvert\ell_f-\ell_i+1\rvert \leqslant \lambda$ $\Delta\ell \leqslant \lambda+1$ $\ell_i+\ell_f+\lambda$ odd
$\langle\lVert T_{\lambda L}(\vec{r},\vec{\sigma}\times\vec{p})\rVert\rangle$	$(-1)^{L+1}$			$\Delta N \leqslant L-1$ $\lvert N_f-N_i+1\rvert \leqslant L$ $\Delta N \leqslant L+1$	$\Delta\ell \leqslant L-1$ $\lvert\ell_f-\ell_i+1\rvert \leqslant L$ $\Delta\ell \leqslant L+1$ $\ell_i+\ell_f+L$ odd

Table 5. Odd-A Nuclei - Selection Rules for Λ & Σ

Matrix Element	Λ, Σ		
$\langle \| Y_\lambda(\vec{r}) \| \rangle$	$k=\Lambda_f-\Lambda_i$ $q=-\Lambda_f-\Lambda_i$	for	$\Sigma_i=\Sigma_f$ $\Sigma_i=-\Sigma_f$
$\langle \| T_{\lambda L}(\vec{r},\vec{\sigma}) \| \rangle$	$k=\Lambda_f-\Lambda_i$ $=\Lambda_f-\Lambda_i\mp 1$ $q=-\Lambda_f-\Lambda_i$ $=-\Lambda_f-\Lambda_i\mp 1$	for	$\Sigma_i=\Sigma_f$ $\Sigma_i=\pm 1/2 \quad \Sigma_f=\mp 1/2$ $\Sigma_i=-\Sigma_f$ $\Sigma_i=\Sigma_f=\pm 1/2$
$\langle \| T_{\lambda L}(\vec{r},\vec{p}) \| \rangle$	$k=\Lambda_f-\Lambda_i$ $q=-\Lambda_f-\Lambda_i$	for	$\Sigma_i=\Sigma_f$ $\Sigma_i=-\Sigma_f$
$\langle \| Y_\lambda(\vec{r})\vec{\sigma}\cdot\vec{p} \| \rangle$	$k=\Lambda_f-\Lambda_i$ $=\Lambda_f-\Lambda_i\mp 1$ $q=-\Lambda_f-\Lambda_i$ $=-\Lambda_f-\Lambda_i\mp 1$	for	$\Sigma_i=\Sigma_f$ $\Sigma_i=\pm 1 \; 2 \quad \Sigma_f=\mp 1/2$ $\Sigma_i=-\Sigma_f$ $\Sigma_i=\Sigma_f=\pm 1/2$
$\langle \| T_{\lambda L}(\vec{r},\vec{\sigma}\times\vec{p}) \| \rangle$	$k=\Lambda_f-\Lambda_i$ $=\Lambda_f-\Lambda_i\mp 1$ $q=-\Lambda_f-\Lambda_i$ $=-\Lambda_f-\Lambda_i\mp 1$	for	$\Sigma_i=\Sigma_f$ $\Sigma_i=\pm 1 \; 2 \quad \Sigma_f=\mp 1 \; 2$ $\Sigma_i=-\Sigma_f$ $\Sigma_i=\Sigma_f=\pm 1/2$

Table 6. Odd-odd Nuclei - Selection Rules for Λ & Σ .
Transition Between States of the Same Relative Coupling

$$\Omega_i' - \Omega_i \rightarrow \Omega_f' - \Omega_f \qquad \Omega_i' = \Omega_f'$$

Matrix Element	Λ & Σ		
$\langle \| Y_\lambda(\vec{r}) \| \rangle$	$k = -\Lambda_f + \Lambda_i$	for	$\Sigma_i = \Sigma_f$
	$k = -\Lambda_f + \Lambda_i$	for	$\Sigma_i = \Sigma_f$
$\langle \| T_{\lambda L}(\vec{r},\vec{\sigma}) \| \rangle$	$= -\Lambda_f + \Lambda_i - 1$		$\Sigma_i = -1/2,\ \Sigma_f = 1/2$
	$= -\Lambda_f + \Lambda_i + 1$		$\Sigma_i = 1/2,\ \Sigma_f = -1/2$
$\langle \| T_{\lambda L}(\vec{r},\vec{p}) \| \rangle$	$k = -\Lambda_f + \Lambda_i$	for	$\Sigma_i = \Sigma_f$
	$k = -\Lambda_f + \Lambda_i$	for	$\Sigma_i = \Sigma_f$
$\langle \| Y_\lambda(\vec{r})\vec{\sigma}\cdot\vec{p} \| \rangle$	$= \Lambda_f + \Lambda_i - 1$		$\Sigma_i = -1/2,\ \Sigma_f = 1/2$
	$= \Lambda_f + \Lambda_i + 1$		$\Sigma_i = 1/2,\ \Sigma_f = -1/2$
$\langle \| T_{\lambda L}(\vec{r},\vec{\sigma} \times \vec{p}) \| \rangle$	$k = -\Lambda_f + \Lambda_i$	for	$\Sigma_i = \Sigma_f$
	$= -\Lambda_f + \Lambda_i - 1$		$\Sigma_i = -1/2,\ \Sigma_f = 1/2$
	$= -\Lambda_f + \Lambda_i + 1$		$\Sigma_i = 1/2,\ \Sigma_f = -1/2$

For $\rho_i = \rho_f$, make the change: $k \rightarrow q,\ N_f, \Lambda_f, \Sigma_f \rightarrow N_f', \Lambda_f', \Sigma_f'$

Table 7. Odd-odd Nuclei - Selection Rules for Λ & Σ.
Transition Between States of Different Relative Couplings.

$$\Omega_i'+\Omega_i \rightarrow \Omega_f'-\Omega_f \quad \Omega_i'=\Omega_f'$$

Matrix Elements	Λ & Σ		
$\langle \| Y_\lambda(\vec{r}) \| \rangle$	$k = -\Lambda_f-\Lambda_i$	for	$\Sigma_i = -\Sigma_f$
$\langle \| T_{\lambda L}(\vec{r},\vec{\sigma}) \| \rangle$	$k = -\Lambda_f-\Lambda_i$ $= -\Lambda_f-\Lambda_i \mp 1$	for	$\Sigma_i = -\Sigma_f$ $\Sigma_i = \Sigma_f = \pm 1/2$
$\langle \| T_{\lambda L}(\vec{r},\vec{p}) \| \rangle$	$k = -\Lambda_f-\Lambda_i$	for	$\Sigma_i = -\Sigma_f$
$\langle \| \sigma Y_\lambda(\vec{r})\vec{\sigma}\cdot\vec{p} \| \rangle$	$k = -\Lambda_f-\Lambda_i$ $= -\Lambda_f-\Lambda_i \mp 1$	for	$\Sigma_i = -\Sigma_f$ $\Sigma_i = \Sigma_f = \pm 1/2$
$\langle \| T_{\lambda L}(\vec{r},\vec{\sigma} \times \vec{p}) \| \rangle$	$k = -\Lambda_f-\Lambda_i$ $= -\Lambda_f-\Lambda_i \mp 1$	for	$\Sigma_i = -\Sigma_f$ $\Sigma_i = \Sigma_f = \pm 1/2$
For $\Omega_i'=\Omega_f$, make the change: $k \rightarrow q$, $N_f, \ell_f, \Lambda_f, \Sigma_f \rightarrow N_f', \ell_f', \Lambda_f', \Sigma_f'$			

CHAPTER IX

NUCLEAR PARAMETERS FOR THE FIRST-FORBIDDEN BETA DECAY

In this chapter, the expressions given in Chapter VII for the nuclear matrix elements for even-mass nuclei will be used to determine the nuclear parameters for first-forbidden beta decay with spin change $|\Delta J| = 1$. The nuclear matrix elements involved in this situation are

$$\int \vec{r} \quad , \quad \int i\vec{\alpha} \quad , \quad \int i\vec{\sigma}\times\vec{r} \quad \text{and} \quad \int B_{ij}$$

in the Konopinski and Uhlenbeck notation or

$$\langle f \| Y_{j1}(\vec{r}) \| i \rangle \quad , \quad \langle f \| T_{10}(\vec{r},\vec{p}) \| i \rangle \quad , \quad \langle f \| T_{11}(\vec{r},\vec{\sigma}) \| i \rangle \text{ and } \langle f \| T_{21}(\vec{r},\vec{\sigma}) \| i \rangle$$

in the spherical tensor notation. The correspondence between these two notations is given in Table 1 (Chapter III).

With the exception of the log ft-value, all theoretical expressions for experimental observables in first-forbidden beta decay are written not in terms of the nuclear matrix elements, but in terms of their ratios. Thus, for the first-forbidden beta decay with spin change $|\Delta J| = 1$ the following nuclear parameters are introduced by Kotani[21].

$$\eta x = -C_V \int \vec{r} \quad , \quad \eta u = C_A \int i\vec{\sigma}\times\vec{r} \quad , \tag{9-1a}$$

$$\eta \xi' y = -C_V \int i\vec{\alpha} \quad , \quad \eta z = C_A \int B_{ij} \quad . \tag{9-1b}$$

In the present calculation the standard matrix element is chosen such that

$$\alpha = 1 \qquad \text{and} \qquad \eta = C_A \int B_{ij}\,^{*} , \tag{9-2}$$

and the following parameters are evaluated,

$$x = -\frac{C_V}{C_A}\frac{\int \vec{r}}{\int B_{ij}} \quad , \quad u = \frac{\int i\vec{\sigma}\times\vec{r}}{\int B_{ij}} \quad \text{and} \quad \Lambda = \frac{\xi' y'}{\xi x} = \frac{\int i\vec{\alpha}}{\xi \int \vec{r}} .$$

In the expression for Λ, ξ is a parameter defined by

$$\xi = \frac{\alpha Z}{2\rho}$$

where α is the fine structure constant and ρ is the radius of the nucleus given by

$$\rho = 1.2\, A^{1/3} \times 10^{-13} \text{ cm} .$$

The value of Λ was first predicted theoretically by Ahrens and Feenberg[25] and independently by Pursey[26]. Recently, the conserved vector-current theory in weak interaction was used by Fujita[27] to derive Λ. The proposed values for Λ are

$\Lambda \sim 1$ Ahrens and Feenberg,

$\Lambda \sim 2$ Pursey,

$\Lambda \sim 2.5$ Fujita.

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* The notation η for the matrix element is used only in Equations (9-1) and (9-2). Elsewhere, η denotes the deformation parameter.

In the following, the calculation of nuclear parameters is made for β^- decays in Tm^{170} and Re^{186}. In both isotopes, the transition under consideration takes place from the ground state of an odd-odd nucleus with $J_i=1$ to the first-excited state of an even-even nucleus with $J_f=2$. In both cases the final state is interpreted as being the first excited state in the rotational band of the intrinsic ground state. The pertinent parts of the decay schemes of Tm^{170} and Re^{186} are shown in Figure 2. The accepted values of J, K and π, the parity, are indicated for each level.

As discussed in Chapter V, two-particle intrinsic wave functions for both the initial and final states should adequately describe the situation for the transition under consideration. The intrinsic state for the even-even nucleus is described as a proton-neutron state in which the odd proton and odd neutron couple together so that

$$K = \Omega = | \Omega_p \pm \Omega_n | \ .$$

The intrinsic state for the even-even nucleus is described as a proton-proton state in which the two protons pair off in the same Nilsson state so that

$$K = \Omega = 0 \ .$$

During the β^- transition it is assumed that the odd proton remains in the same Nilsson state while the odd neutron is transformed into a proton that pairs off in the same Nilsson state with the odd proton. Therefore the equations presented in Chapter VII for the beta decay nuclear matrix elements for even-mass nuclei using two-particle intrinsic wave functions should apply to the transitions under consideration.

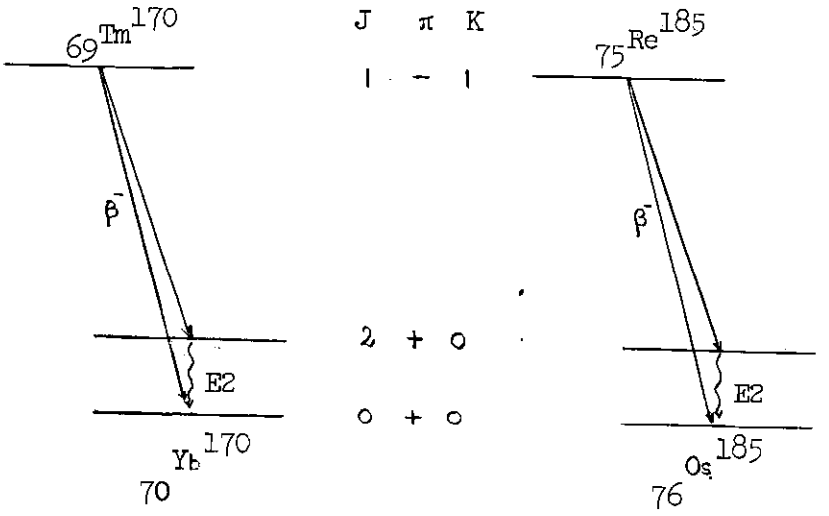

Figure 2. Decay Scheme of Tm^{170} and Re^{185}.

For each nucleus the deformation parameter η(or $\delta \sim 0.05\eta$) can be estimated from Figure 9 in Mottelson and Nilsson's paper[22]. In the figure values of δ determined from experimental E2 transitions between two states in a rotational band are plotted as a function of the mass number. For a fixed value of η (or δ) the intrinsic wave function for a nucleus is unique. According to Figure 9[22] the deformation parameters η are approximately equal to 6 and 4 for Tm and Re, respectively. Therefore, the intrinsic wave functions for these isotopes are determined at $\eta=6$ for Tm and at $\eta=4$ for Re from the Nilsson energy level diagrams. However, in order to show the variation of nuclear parameters with respect to the deformation, the calculations are also made for two other values of η besides the values chosen above, with the same base vectors in the expansion of the intrinsic state. It should be pointed out that if the deformation parameter, η, is changed, different intrinsic wave functions may be indicated due to level crossings on the Nilsson diagrams.

In Nilsson's representation, the intrinsic state also depends on the relative strength of of the spin-orbit term and of the ℓ^2-term in the single-particle Hamiltonian. In his numerical calculations, Nilsson has assigned values of μ for each N-shell so as to reproduce (for $\delta=0$) the assumed sequence of shell model levels as well as possible. If the value of μ is changed, the base vectors in the expansion of a particular intrinsic state X_Ω remain the same. However, the expansion coefficients $a_{\ell\Lambda}$ change. For both the Tm^{170} and Re^{186} transitions under consideration the initial intrinsic state of the transforming nucleon belongs to the shell $N_i=5$, and the final intrinsic state

belongs to the shell N=4. For the shell N_i=5 the expansion coefficients are given by Nilsson[17] with μ_i=0.45, and by Nilsson and Mottelson[22] with μ_i=0.7. For the shell N_f =4 the expansion coefficients are calculated by Nilsson[17] with μ =0.55. In order to show the variation of nuclear parameters with respect to the parameter μ, the calculations are made for two sets of μ values: μ_i=0.45, μ_f =0.55 and μ_i=0.70, μ_f =0.55.

After lengthy numerical calculations, the nuclear matrix elements are of the form

$$\langle f \| T_{\lambda L} \| i \rangle = A \sum_{i,f} C_{if}\, a_i\, a_f \,. \qquad (9\text{-}4)$$

The numerical values of the constants A and C_{if} will be given in a table for each isotope. The values of the expansion coefficients "a" may be found in papers of Nilsson[17], and Nilsson and Mottelson[22]. The variation of nuclear parameters x, u, and Λ with respect to η and μ are given in Tables 12 and 13, respectively.

For convenience, in the next two sections the intrinsic state will be written either as $\Omega\pi[Nn_z\Lambda\Sigma]$ (the asymptotic wave function for very large deformation) or as $|N\Omega\alpha\rangle$ (the actual intrinsic wave function X_Ω, where α is the level number in the Nilsson diagrams). The correspondence between $[Nn_z\Lambda\Sigma]$ and $|N\Omega\alpha\rangle$ can be established easily and may be found in Nilsson's paper[17].

Tm^{170}

The decay scheme of Tm^{170} is shown in Figure 2. The estimated value of the deformation parameter η of Tm^{170} is approximately equal to 6. In figures 3 and 4 of reference[22] Mottelson and Nilsson have

plotted the single-particle proton energy levels for $50 < Z < 82$ and the single-particle neutron energy levels for $82 < N < 126$. For $\eta=6$ the 69th proton (the odd proton for Tm^{170}) is apparently in the Nilsson state $1/2 + [411\downarrow]$. However for $\eta=6$, three different neutron levels are fairly close together in the neighborhood of the 101st neutron. From the spin $J=1$ (and hence $K=\Omega=1$) and the measured magnetic moment, Gallagher and Soloviev[23] concluded that for the ground state of Tm^{170} the odd neutron is in the Nilsson state $1/2-[521\downarrow]$. Therefore, the initial intrinsic state are taken as $1/2+[411\downarrow]$ and $1/2-[521\downarrow]$ for the proton and the neutron, respectively. In the final state, both protons are characterized by $1/2+[411\downarrow]$. In order to meet the requirement $K=\Omega$, it is necessary that

$$\Omega = \Omega_p + \Omega_n = \tfrac{1}{2} + \tfrac{1}{2}$$

for the initial state with $K_i=1$, and

$$\Omega = \Omega_p - \Omega_p = \tfrac{1}{2} - \tfrac{1}{2}$$

for the final state with $K_f = 0$. The two-particle intrinsic wave functions are symbolically written as

$$|i\rangle = [411\downarrow] + [521\downarrow] ,$$

$$|f\rangle = [411\downarrow] - [411\downarrow] .$$

Thus, the transition takes place between states of different relative coupling, i.e.,

$$\Omega_{p_i} + \Omega_{n_i} \longrightarrow \Omega_{p_f} - \Omega_{p_f}$$

in the notation of Chapter VII, or more specifically

$$\Omega_p + \Omega_n \longrightarrow \Omega_p - \Omega_p \cdot$$

Therefore, for the Tm^{170} transition, the general expression for the reduced matrix elements is given by Equation (7-12). The intrinsic wave functions for the transforming nucleon are

$$|i\rangle|5\tfrac{1}{2}63\rangle = a_{50}|550+\rangle + a_{30}|530+\rangle + a_{10}|510+\rangle$$

$$+ a_{51}|551-\rangle + a_{31}|531-\rangle + a_{11}|511-\rangle , \qquad (9\text{-}5)$$

and

$$|f\rangle = |4\tfrac{1}{2}43\rangle = a_{40}|440+\rangle + a_{20}|420+\rangle + a_{00}|400+\rangle$$

$$+ a_{41}|441-\rangle + a_{21}|421-\rangle \cdot \qquad (9\text{-}6)$$

The coefficients A and C_{if} in Equation (9-4) are given in Table 8. The numerical values of the matrix elements $\int \vec{r}$, $\int i\vec{\alpha}$, $\int i\vec{\sigma} \times \vec{r}$ and $\int B_{ij}$ are given in Table 9. The parameters x, u and Λ are given in Tables 12 and 13. In Table 12 values of the matrix element parameters are also presented for η equal to 4 and 2 in order to show their variation with respect to the deformation. For all of the values of η the same intrinsic states are used, but with different values of the expansion coefficients.

Re^{186}

The decay scheme of Re^{186} is shown in Figure 2. The deformation parameter, η, for Re^{186} is estimated as $\eta=4$. An examination of Figures 3 and 4 of reference[22] indicates that for $\eta=4$ there are two or three closely spaced Nilsson levels in the neighborhood of both the 75th proton and the 111th neutron. However, the choice of intrinsic states is restricted by the fact that for the Re^{186} ground state the parity is odd and $J=K=\Omega=1$. This means that the proton and neutron intrinsic state must have opposite parity and couple to produce $\Omega=1$. Gallagher and Soloviev[23] have included that for the Re^{186} ground state the intrinsic wave functions are clearly established as $\frac{5}{2}+[402\uparrow]$ and $\frac{3}{2}-[512\downarrow]$ for the proton and neutron, respectively. In the final state, both protons are in $1/2+[402\uparrow]$ state. The two-particle wave functions are then written, symbolically,

$$|i\rangle = [402\uparrow] - [512\downarrow],$$

$$|f\rangle = [402\uparrow] - [402\downarrow].$$

Therefore, the transition takes place between states of the same relative coupling, i.e.,

$$\Omega_i' - \Omega_i \rightarrow \Omega_f - \Omega_f$$

or more specifically,

$$\Omega_p - \Omega_n \rightarrow \Omega_p - \Omega_p$$

The intrinsic wave functions for the transforming nucleon are

$$|i\rangle = |5\tfrac{3}{2}\,62\rangle = a_{51}|551+\rangle + a_{31}|531+\rangle + a_{11}|511+\rangle$$

$$+ a_{52}|552-\rangle + a_{32}|532-\rangle\,, \qquad (9\text{-}9)$$

and

$$|f\rangle = |4\tfrac{5}{2}\,31\rangle = a_{42}|442+\rangle + a_{22}|422+\rangle + a_{43}|443-\rangle\,. \qquad (9\text{-}10)$$

The coefficients A and C_{if} in Equation (9-4) are given in Table 10. The numerical values of the matrix elements $\int\vec{r}$, $\int i\vec{\alpha}$, $\int i\vec{\sigma} \times \vec{r}$ and $\int$Bij are given in Table 11. The variation of the nuclear parameters x, u and Λ, with respect to the deformation, is illustrated in Tables 12 and 13.

Table 8. Coefficients A and C_{if} for Tm^{170}

	C_{if}			
$a_i a_f$	$\langle f\|\|Y_1(\vec{r})\|\|i\rangle$	$\langle f\|\|T_{10}(\vec{r},\vec{\sigma})\|\|i\rangle$	$\langle f\|\|T_{11}(\vec{r},\vec{\sigma})\|\|i\rangle$	$\langle f\|\|T_{21}(\vec{r},\vec{\sigma})\|\|i\rangle$
$a_{50}a_{41}$	0.745	0.745	0.745	0.745
$a_{30}a_{41}$	-0.398	-1.195	-0.398	-0.398
$a_{30}a_{21}$	0.621	0.621	0.621	0.621
$a_{10}a_{21}$	-0.632	-1.897	-0.632	-0.632
$a_{51}a_{40}$	0.913	0.913	-0.913	-0.913
$a_{31}a_{40}$	-0.309	-0.926	0.309	0.309
$a_{31}a_{20}$	0.878	0.878	-0.878	-0.878
$a_{11}a_{20}$	-0.365	-1.095	0.365	0.365
$a_{11}a_{00}$	1.080	1.080	-1.080	-1.080
$a_{50}a_{40}$			1.667	-1.667
$a_{30}a_{40}$			0.713	-0.713
$a_{30}a_{20}$			1.521	-1.521
$a_{10}a_{20}$			1.033	-1.033
$a_{10}a_{00}$			1.527	-1.527
A	$-\left(\frac{3}{40\pi}\right)^{1/2}$	$-\frac{i}{4\pi}\left(\frac{3}{10}\right)^{1/2}$	$-\frac{3}{8\pi}\left(\frac{1}{5}\right)^{1/2}$	$-\frac{3}{8\pi}\left(\frac{3}{5}\right)^{1/2}$

Table 9. Matrix Elements for Tm^{170}
(μ_i=0.45, μ_f=0.55)

Matrix Elements in units of $\frac{1}{\sqrt{10}}(\frac{\hbar}{M\omega_o})^n$	η=2	η=4	η=6
$\int \vec{r}$, n=1/2	0.023	0.026	0.018
$i\int \alpha$, n=-1/2	0.612 $\frac{1}{Mc}$	0.486 $\frac{1}{Mc}$	0.400 $\frac{1}{Mc}$
$\int \sigma$ x r, n=-1/2	-1.116	-0.892	-0.705
$\int B_{ij}$, n=1/2	0.348	0.218	0.463

Table 10. Coefficients A and C_{if} for Re^{186}

	C_{if}			
$a_i a_f$	$\langle f \| Y_1(\vec{r}) \| i \rangle$	$\langle f \| T_{10}(\vec{r},\vec{p}) \| i \rangle$	$\langle f \| T_{11}(\vec{r},\vec{\sigma}) \| i \rangle$	$\langle f \| T_{21}(\vec{r},\vec{\sigma}) \| i \rangle$
$a_{51}a_{42}$	0.577	0.577	-0.577	-0.577
$a_{31}a_{42}$	-0.488	-1.464	0.488	0.488
$a_{31}a_{22}$	0.359	0.359	-0.359	-0.359
$a_{11}a_{22}$	-0.894	-2.683	0.894	0.894
$a_{52}a_{43}$	0.408	0.408	0.408	0.408
$a_{32}a_{43}$	-0.577	-1.732	-0.577	-0.577
$a_{52}a_{42}$			1.527	-1.527
$a_{32}a_{42}$			0.617	-0.617
$a_{32}a_{22}$			1.134	-1.134
A	$-\left(\frac{3}{40\pi}\right)^{1/2}$	$-\frac{i}{4\pi}\left(\frac{3}{10}\right)^{1/2}$	$-\frac{3}{8\pi}\left(\frac{1}{5}\right)^{1/2}$	$-\frac{3}{8\pi}\left(\frac{3}{5}\right)^{1/2}$

Table 11. Matrix Elements for Re^{186}
(μ_i=0.45, μ_f=0.55)

Matrix Elements in unit of $\frac{1}{\sqrt{10}}(\frac{\hbar}{M\omega_o})^n$	η=2	η=4	η=6
$\int \vec{r}$, n=1/2	-0.151	-0.101	-0.079
$-i\int \vec{\alpha}$, n=-1/2	-0.700 $\frac{1}{Mc}$	-0.507 $\frac{1}{Mc}$	-0.410 $\frac{1}{Mc}$
$\int \vec{\sigma} \times \vec{r}$, n=1/2	-0.553	-0.652	-0.714
$\int B_{ij}$, n=1/2	2.927	2.922	2.890

Table 12. Nuclear Parameters for Tm^{170} and Re^{186} as a Function of η (μ_i=0.45, μ_f=0.55)

	η	2	4	6
Tm^{170}	x	0.055	-0.101	-0.032
	u	-3.208	4.185	1.523
	Λ	25.968	18.108	21.098
Re^{186}	x	-0.043	-0.029	-0.023
	u	-0.189	-0.223	-0.247
	Λ	4.166	4.505	4.662

Table 13. Nuclear Parameters for Tm^{170} and Re^{186} as a Function of μ (μ_f=0.55)

	η	μ_i	x	u	Λ
Tm^{170}	6	0.45	-0.032	1.523	21.098
		0.70	-0.353*	2.135	0.332
Re^{186}	4	0.45	-0.029	-0.223	4.505
		0.70	0.004	0.293	2.375

*The values of x and u here are equivalent to those obtained by Bertheir and Lipnik - "equivalent" since different standard matrix elements are used.

CHAPTER X

CONCLUSIONS

The derivation of nuclear matrix elements in beta decay was made previously by Bogdan[16], using the Nilsson single-particle model. The expressions so obtained were then used to calculate the nuclear matrix elements for the first-forbidden beta decay of Tm^{170} and Re^{186}. There is a difference in signs of $k=K_f - K_i$ and $q=-K_f - K_i$ in the intrinsic part between Bogdan's expressions and the expressions given in the present dissertation, using the one-particle model. As a result of this sign, all the nuclear matrix elements for the decay of Tm^{170} vanish while non-zero results are obtained for the decay of Re^{186}. If the expressions for nuclear matrix elements, Equation (6-7) or (6-8), derived here were used, the opposite results would be obtained. However, the two-particle wave function discussed later also yields nonvanishing results for Re^{186}.

When this dissertation was in its final form, a preprint was received from Berthier and Lipnik[28]. In their paper the derivation of nuclear matrix elements for non-relativistic operators in β^- decay is made. For odd A-odd N nuclei one-particle intrinsic wave functions are used, and the expression obtained by Berthier and Lipnik is identical with Equation (6-7) in this thesis. For odd-odd nuclei, two-particle intrinsic wave functions are used, and their expression, Equation (14), is also the same as Equation (7-6) given here for the transition between states of different relative couplings. They then use Equation (14) to

evaluate the nuclear parameters for Tm^{170}. It is found that the results obtained by Berthier and Lipnik are essentially equivalent to the results given in the present dissertation.*

For strongly deformed nuclei, the wave function of the form $\psi \omega D^{J}_{MK} X_{\Omega}$, Equations (2-34) and (2-37), is a good representation of nuclear states. Hence, the rotational motion contributes but a geometrical factor in the reduced matrix element through the C-coefficients $\langle J_i \lambda K_i k \mid J_f K_f \rangle$ and $\langle J_i \lambda K_i q \mid J_f - K_f \rangle$. These C-coefficients will yield a well-defined and non-zero factor provided the triangle condition $\Delta(J_i \lambda J_f)$ is satisfied, and $|k|$ and/or $|q|$ are not larger than the rank λ of the tensor operator in question. The case $|k| > \lambda$ (then one also has $|q| > \lambda$ since $|q| \geq |k|$) will lead to the K-forbiddenness. Thus, because of the K-forbiddenness all the nuclear matrix elements in the decay of Eu vanish. When such a case arises, as was suggested by Alaga *et al.*[29], one would need small perturbation terms which allow the contribution of more than one K-value. The effect of these perturbation terms might well be small, however it will become significant when the selection rules for the K's rule out all other possibilities.

Aside from the dependence on the rotational motion discussed above, the reduced matrix elements depend entirely upon the intrinsic structure of the nucleus via the intrinsic matrix elements. This dependence can be

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* The results are equivalent since different standard matrix elements are used. The standard matrix element is chosen by Berthier and Lipnik such that $u'=1$ and $\eta'=C_A \int i\vec{\sigma} \times \vec{r}$. Hence, the numerical values of their nuclear parameters, x' and z', are related to those given in Table 13 by $x'=xu^{-1}$ and $z'=u^{-1}$, respectively.

divided into two different kinds: the arrangement of the last few nucleons through the coupling of their angular momenta, and the deformation of the nucleus. This will be summarized in the next two sections of this chapter.

The assumption that the nucleus possesses a symmetry axis implies

$$K = \Omega = \sum_i \Omega_i \ ,$$

where K, Ω and Ω_i are the projection on the symmetry axis of the total nuclear spin $\vec{J}$, of the total angular momentum $\vec{j}$ of the nucleons and of the angular momentum $\vec{j}_i$ of the ith nucleon, respectively. Since Ω_i is always a half odd-integer and since K is a half odd-integer for odd-A nuclei and is an integer for even-A nuclei, Equations (5-11) and and (5-12), it is necessary that an odd (even) number of intrinsic wave functions, X_{Ω_i}, for individual nucleons be used in the construction of the intrinsic wave function X_Ω for an odd-A (even-A) nucleus. The signs in the sum $\Omega=\sum_i \Omega_i$ may be + or - . If the relative signs are the same between the initial and final states, the transition is called the transition between states of the same relative coupling. Otherwise, one has the transition between states of different relative couplings[18].

For odd-A nuclei, one-particle intrinsic wave functions are used in the present thesis, hence there will be no problem and the expression so obtained for the nuclear matrix elements is unique. This, of course, may be applied to the β^- decay of an odd A-odd N nucleus or to the β^+ decay of an odd A-odd Z nucleus. For the $\beta^-(\beta^+)$ decay of an odd A-odd Z

(-odd N) nucleus, it is desirable that three-particle intrinsic wave functions be used. Although this situation is not considered in this dissertation, it is safe to say that if the projections Ω_i of the like nucleons (protons or neutrons) cancel by pairs so that the total nuclear spin is determined by the Ω-value of the last odd nucleon, then the transition always takes place between states of the same relative coupling. Hence, the expression for the nuclear matrix element is still unique. For the β^- decay of an odd A-odd Z nucleus for example, one may write symbolically

$$\Omega_n - \Omega_n + \Omega_p \longrightarrow \Omega_p - \Omega_p + \Omega_n .$$

For even-A nuclei, two-particle intrinsic wave functions are used. The transitions between states of the same relative couplings and between states of different relative couplings lead to different sets of selection rules. The two modes of transition are illustrated in the present dissertation by the evaluations of nuclear parameters for the isotope of Tm^{170} and Re^{186}. For Tm^{170} the transition is classified as a transition between states of different relative couplings, which is symbolically written as

$$\Omega_p + \Omega_n \longrightarrow \Omega_p - \Omega_p \quad ;$$

while for Re^{186}, it is a transition between states of the same relative coupling, i.e.,

$$\Omega_p - \Omega_n \longrightarrow \Omega_p - \Omega_p \quad .$$

If the same expression were used in both cases, the nuclear parameters of one or the other isotope would vanish identically because of the selection rules. For this reason, Expression (14) given by Berthiér and Lipnik would yield net zero results if applied to Re^{186}.

The Nilsson model depends on four parameters: the frequency, ω_o, the deformation parameter, η(or δ), and the strength of the spin-orbit terms, C, and of the ℓ^2-term, D. For heavy nuclei the reasonable value for ω_o is given by $\hbar\omega_o(0)=41A^{-1/3}$ Mev($\hbar\omega_o(0)\sim 8.8$ Mev for $A\sim 100$). In his calculation, Nilsson[17] used the value $\kappa = -0.05$ to determine the energy level, hence the strength C of the spin-orbit term is fixed by Equation (2-53).

For each nucleus the deformation parameter η is given by Figure 9 in Mottelson and Nilsson's paper[22] and the appropriate intrinsic wave function is chosen at this η-value according to the Nilsson energy level diagrams. Hence, in principle, the intrinsic wave function is unique for each nucleus. For the isotope of Tm^{170} and Re^{186}, the intrinsic wave functions are chosen at $\eta=6$ and $\eta=4$, respectively. However, in order to show the variation of the nuclear parameters with respect to η, the calculations were made for $\eta=2,4,6$ using the same base vectors $|N\ell\Lambda\Sigma\rangle$ in the expansion of the intrinsic state, X_Ω. The results show that the nuclear parameters change smoothly as η varies for Re^{186} but not for Tm^{170} as shown in Table 12. It should be noted that for such a variation in η, another intrinsic wave function might have to be chosen to appropriate the correct energy level at a level crossing.

Finally, the intrinsic wave function depends on the relative strength, μ, of the spin-orbit term and of the ℓ^2-term as defined by Equation (2-54). For each shell N, a particular value is taken for μ. However, if another μ-value is chosen, the base vectors $|N\ell\Lambda\Sigma\rangle$ in the expansion of a particular intrinsic state, X_Ω, remain unchanged. The calculations are performed for two sets of μ-values: μ_i=0.45, μ_f =0.55 and μ_i=0.70, μ_f = 0.55, with η=6 and η=4 for Tm^{170} and Re^{186}, respectively. The results are given in Table 13.

The nuclear parameters for Tm^{170} are found to be very sensitive to the change of the relative strength, μ, of the spin-orbit term and of the ℓ^2-term, and to the change of the deformation parameter η in the Nilsson model. Besides, the nuclear parameters so obtained do not seem to agree with experiments[30]. The numerical values of the parameter x are reasonably small but the values of μ are consistently too large. For μ_i=0.45, μ_f=0.55 the parameter Λ is definitely beyond the reach of all theoretical values predicted by Ahrens and Feenberg[25], Pursey[26], and Fujita[27]. For μ_i=0.70, μ_f=0.55 the value obtained for Λ is rather small. Due to this sensitivity, it is hoped that a close study of the isotope of Tm^{170} would yield certain understanding of the intrinsic structure of the nucleus, in particular of the Nilsson model with its ℓ^2-term which, so far, has no theoretical justification.

Unlike Tm^{170}, the results obtained for Re^{186} are consistent as can be seen in Tables 12 and 13. The calculated values of the nuclear parameters for Re^{186} are within the range of permissible values, in agreement

with experiment[30]. For μ_i=0.45 the value of Λ is rather large compared with Λ predicted by Fujita[27]. For μ_i=0.70 these Λ-values are in good agreement. For comparison, two graphs, Figures 3 and 4, obtained by Dulaney et al.[30] are reproduced here. Figure 4 shows that for small values of x and u all the graphs corresponding to different Λ-values converge to the same region. In other words, x and u do not depend on Λ so long as they are reasonably small.

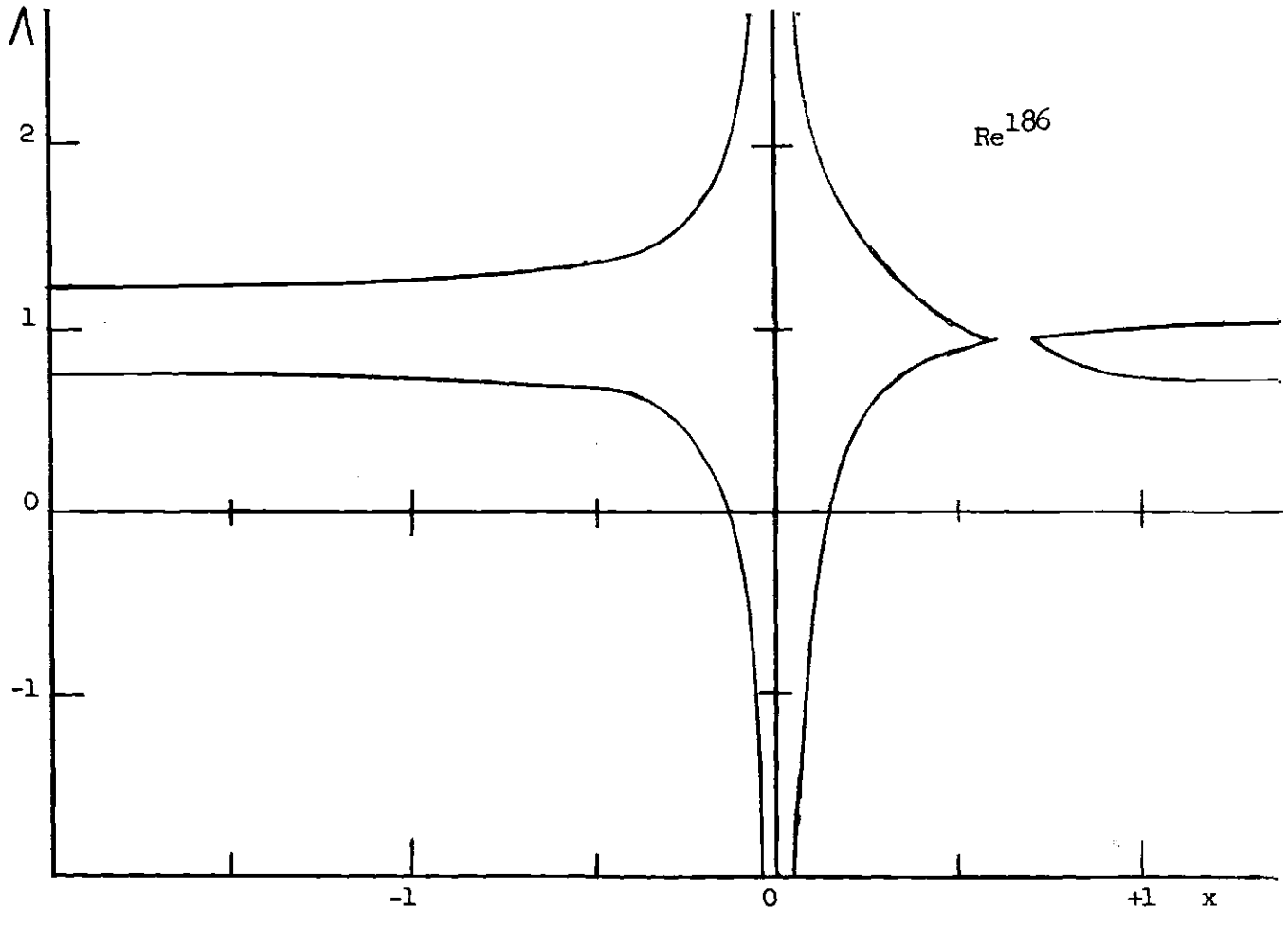

Figure 3. Permissible Range of Λ Values Plotted as a Function of x for Re^{186}.

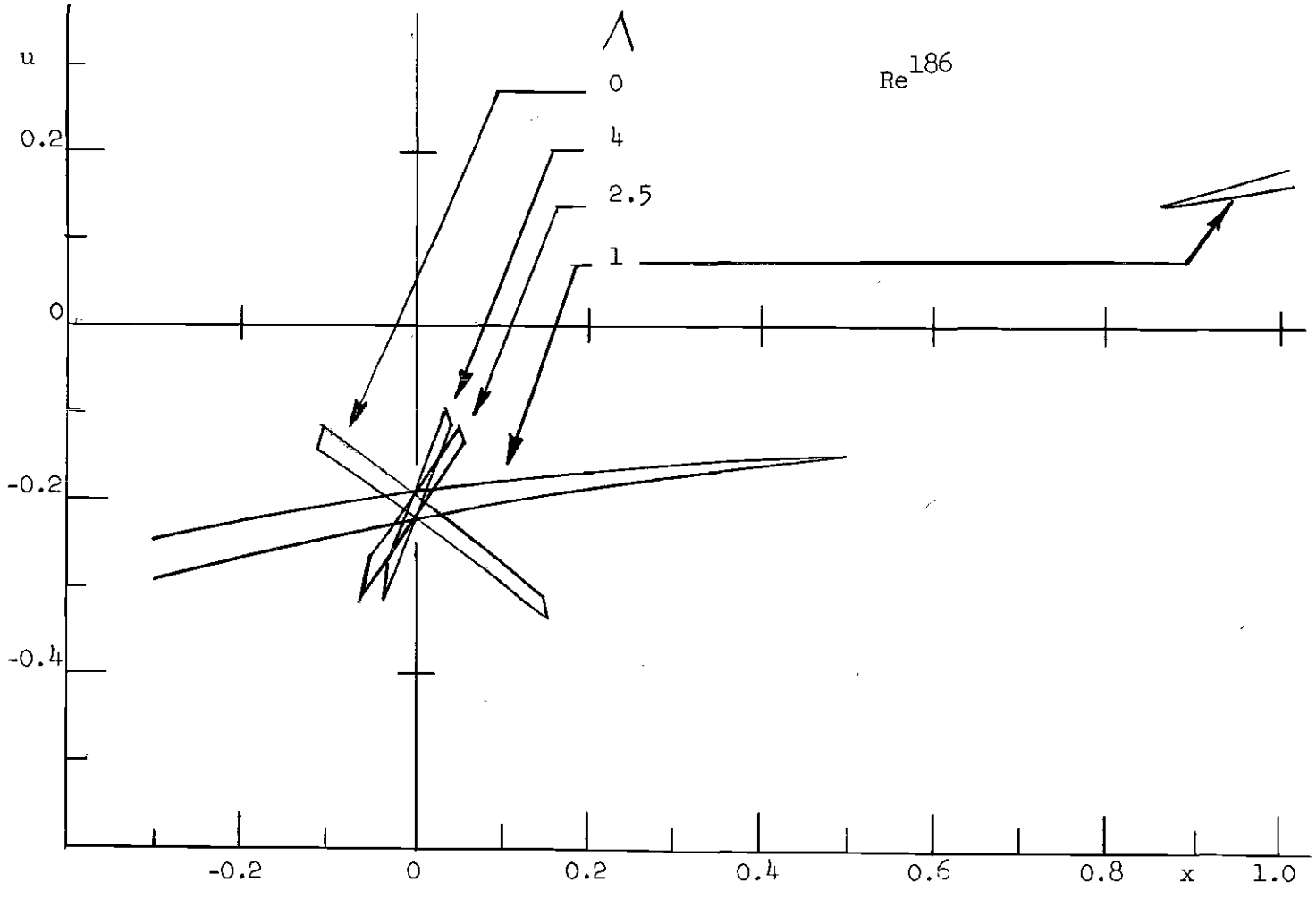

Figure 4. Suitable Regions of the x, u Plane Indicated for Various Values of Λ for Re^{186}.

APPENDIX I

Throughout this dissertation, the algebra of angular momentum was extensively used. This appendix will provide sufficient information concerning that algebra, together with various notations and conventions. Detailed treatments can be found in a number of standard references.[31]

Coupling of Two Angular Momenta - C-coefficients

Consider two commuting angular momentum operators $\vec{J}_i$ $(i=1,2)$, satisfying*

$$J_{iz}|j_i m_i\rangle = m_i |j_i m_i\rangle \,, \tag{I-1}$$

$$J_i^2 |j_i m_i\rangle = j_i(j_i+1)|j_i m_i\rangle \,. \tag{I-2}$$

Define

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \,. \tag{I-3}$$

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* The following notations are used throughout the calculation. A state ψ_α characterized by the quantum number α is represented by the ket-vector $|\psi_\alpha\rangle$ or simply $|\alpha\rangle$. The Hermitian conjugate of the state ψ_α is represented by the bra-vector $\langle\psi_\alpha|$ or $\langle\alpha|$.

The following scalar products are equivalent:

$$\langle\alpha|\alpha\rangle \quad \text{and} \quad \int \psi_\alpha^* \psi_\alpha \, d\tau \,,$$

where the integral is carried over all arguments in ψ_α. Hence:

$$\langle D^{J_3}_{M_3M_3'} | D^{J_2}_{M_2M_2'} | D^{J_1}_{M_1M_1'}\rangle = \int D^{*J_3}_{M_3M_3'} D^{J_2}_{M_2M_2'} D^{J_1}_{M_1M_1'} \, d\Theta_i \,, \text{ etc.}$$

It is obvious that in the direct-product representation $|j_1 m_1\rangle|j_2 m_2\rangle$, J_z is diagonal but J^2 is not diagonal. One can, however, find a new basis $|JM\rangle$ in the $(2j_1+1)(2j_2+1)$ - dimensional linear manifold of product functions in such a way that J^2 and J_z are simultaneously diagonal. This new basis $|JM\rangle$ is defined by

$$|JM\rangle = \sum_{m_1, m_2} \langle j_1 j_2 m_1 m_2 | JM\rangle |j_1 m_1\rangle |j_2 m_2\rangle . \qquad \text{(I-4)}$$

The expansion coefficient $\langle j_1 j_2 m_1 m_2 | JM\rangle$ has different many-names: the Clebsch-Gordon coefficient, the vector-coupling coefficient, the Wigner coefficient, the C-coefficient. For simplicity, it will be called the C-coefficient in this dissertation. Its phase is arbitrary. Here, the phase is chosen such that C is real. It can be shown that $\langle j_1 j_2 m_1 m_2 | JM\rangle$ vanishes unless

$$m_1 + m_2 = M , \quad \text{and} \qquad \text{(I-5a)}$$

$$j_1, j_2, J \text{ satisfy } \Delta(j_1 j_2 J) . \qquad \text{(I-5b)}$$

The notation $\Delta(j_1 j_2 J)$ is called the triangle condition, i.e., j_1, j_2 and J aatisfy the same inequalities as in a triangle. The constraint (I-5a) reduces the sum over the two indices m_1 and m_2 in Equation (I-4) to the sum over only one index, either m_1 or m_2. Throughout the calculation, when the C-coefficient is encountered, it will be understood that the conditions (I-5a) and (I-5b) are satisfied automatically in order to avoid writing all the Kronecker deltas.

The C-coefficients form the matrix of transformation, C. Since

this matrix connects two orthonormal bases in the same linear space, it must be unitary. And since the C-coefficient is real, C is an orthogonal matrix. The orthogonality properties follow immediately.

$$\sum_{m_1} \langle j_1 j_2 m_1 m_2 | JM \rangle \langle j_1 j_2 m_1 m_2 | J'M' \rangle = \delta_{JJ'} \delta_{MM'} , \tag{I-6a}$$

$$\sum_{J} \langle j_1 j_2 m_1 m_2 | JM \rangle \langle j_1 j_2 m_1' m_2' | JM \rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'} . \tag{I-6b}$$

From the orthogonality property (I-6b) , the inverse expression can be obtained:

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{J} \langle j_1 j_2 m_1 m_2 | JM \rangle |JM\rangle . \tag{I-7}$$

The derivation of the C-coefficient is rather complicated and need not be given here. Wigner gives the following explicit expression for this coefficient:

$$\langle j_1 j_2 m_1 m_2 | j_3 m_3 \rangle = \delta_{m_1+m_2, m_3}$$

$$\times \left[(2j_3+1) \frac{(j_3+j_1-j_2)!\,(j_3-j_1+j_2)!\,(j_1+j_2-j_3)!\,(j_3+m_3)!\,(j_3-m_3)!}{(j_1+j_2+j_3+1)!\,(j_1-m_1)!\,(j_1+m_1)!\,(j_2-m_2)!\,(j_2+m_2)!} \right]^{1/2}$$

$$\times \sum_{\nu} \frac{(-1)^{\nu+j_2+m_2}}{\nu!} \frac{(j_2+j_3+m_1-\nu)!\,(j_1-m_1+\nu)!}{(j_3-j_1+j_2-\nu)!\,(j_3+m_3-\nu)!\,(j_1-j_2-m_3+\nu)!} . \tag{I-8}$$

where ν is all integers such that none of the factorial arguments is negative.

For practical purposes, only a few symmetry relations written below

are needed. These symmetry relations can be derived directly from the explicit expression (I-8).

$$\langle j_1 j_2 m_1 m_2 | J M \rangle = (-1)^{j_1+j_2-J} \langle j_1 j_2 -m_1 -m_2 | J -M \rangle \quad , \tag{I-9a}$$

$$= (-1)^{j_1+j_2-J} \langle j_2 j_1 m_2 m_1 | J M \rangle \quad , \tag{I-9b}$$

$$= (-1)^{j_1-m_1} \left(\frac{2J+1}{2j_2+1}\right)^{1/2} \langle j_1 J m_1 -M | j_2 -m_2 \rangle \, . \tag{I-9c}$$

Other possible permutations among j_1, j_2 and J follow from the symmetry relations (I-9a,b,c). Also, from the symmetry relations (I-9), the following expressions are obtained:

$$\langle j_1 \, 0 \; m_1 \; 0 | j_3 \; m_3 \rangle = \delta_{j_1 j_3} \delta_{m_1 m_3} \tag{I-10a}$$

$$\langle \ell_1 \ell_2 \, 0 \, 0 | \ell_3 \, 0 \rangle = 0 \quad \text{unless} \quad \ell_1+\ell_2+\ell_3 = \text{even integer} \, . \tag{I-10b}$$

Other explicit expressions are:

$$\langle j_1 \; m-1 \; 1 | j_1+1 \; m \rangle = \left[\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right]^{1/2} , \tag{I-11a}$$

$$\langle j_1 \; m \; 0 | j_1+1 \; m \rangle = \left[\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}\right]^{1/2} , \tag{I-11b}$$

$$\langle j_1 \; m+1 \; -1 | j_1+1 \; m \rangle = \left[\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}\right]^{1/2} . \tag{I-11c}$$

$$\langle \ell_1 \ell_2 \, 0 \, 0 | \ell_3 \, 0 \rangle = (-1)^{\frac{1}{2}(\ell_1+\ell_2-\ell_3)} \left(\frac{2\ell_3+1}{\ell_1+\ell_2+\ell_3+1}\right)^{1/2}$$

$$\times \frac{\tau(\ell_1+\ell_2+\ell_3)}{\tau(\ell_1+\ell_2-\ell_3)\,\tau(\ell_1-\ell_2+\ell_3)\,\tau(-\ell_1+\ell_2+\ell_3)} \, , \tag{I-12}$$

where $$\tau(x) = \frac{(\frac{1}{2}x)!}{\sqrt{x!}} \quad ,$$

and $$\ell_1 + \ell_2 + \ell_3 = \text{even integer}.$$

Coupling of Three Angular Momenta-Racah Coefficients

Now consider three commuting angular momentum operators $\vec{J}_i$, satisfying Equations (I-1) and (I-2) with i=1,2,3. In order to construct the simultaneous eigenfunctions of $\vec{J}$ and J_z, where

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 \tag{I-13}$$

it suffices to repeat the method in the preceding section twice. The linear operators $\vec{J}$'s are associative. One can write symbolically

$$\vec{J} = (\vec{J}_1 + \vec{J}_2) + \vec{J}_3 = \vec{J}_{12} + \vec{J}_3 \tag{I-14a}$$

$$\vec{J} = \vec{J}_1 + (\vec{J}_2 + \vec{J}_3) = \vec{J}_1 + \vec{J}_{23} \tag{I-14b}$$

The two modes of operation, Equations (I-14a) and (I-14b), are obviously equivalent. However, they provide two different orthonormal bases, $|J_{12}, JM\rangle$ and $|J_{23}, JM\rangle$, in the same $(2j_1+1)(2j_2+1)(2j_3+1)$-dimensional linear manifold. Consequently, they must be connected by an unitary transformation $R(j_1 j_2 J j_3, J_{12} J_{23})$. Racah defined

$$R(abcd,ef) = [(2e+1)(2f+1)]^{\frac{1}{2}} W(abcd,ef), \tag{I-15}$$

W(abcd,ef) if called the Racah coefficient and given by

$$[(2e+1)(2f+1)]^{\frac{1}{2}} W(abcd,ef) = \sum_{\alpha,\beta} \langle ab\,d\,\beta | e\,d+\beta\rangle \langle ed\,\alpha+\beta\,\delta | c\,\alpha+\beta+\delta\rangle$$

$$\cdot \langle b\,d\,\beta\,\delta | f\,\beta+\delta\rangle \langle af\,\alpha\,\beta+\delta | c\,\alpha+\beta+\delta\rangle \cdot \qquad \text{(I-16)}$$

Other expressions follow from the orthogonality properties of the C-coefficients

$$[(2e+1)(2f+1)]^{\frac{1}{2}} W(abcd,ef) \langle af\,\alpha\,\beta+\delta | e\,\alpha+\beta+\delta\rangle =$$
$$= \sum_{\beta} \langle ab\,\alpha\,\beta | e\,\alpha+\beta\rangle \langle ed\,\alpha+\beta\,\delta | c\,\alpha+\beta+\delta\rangle \langle b\,d\,\beta\,\delta | f\,\beta+\delta\rangle \cdot$$
$$\cdot \langle ab\,\alpha\,\beta | e\,\alpha+\beta\rangle \langle ad\,\alpha+\beta\,\delta | c\,\alpha+\beta+\delta\rangle \qquad \text{(I-17a)}$$

$$= \sum_{f} [(2e+1)(2f+1)]^{\frac{1}{2}} \langle b\,d\,\beta\,\delta | f\,\beta+\delta\rangle \langle af\,\alpha\,\beta+\delta | c\,\alpha+\beta+\delta\rangle W(abcd,ef) \qquad \text{(I-17b)}$$

The symmetry relations of the Racah coefficient can be deduced from those of the C-coefficients. In all, there are 24 possible permutations of the six arguments a,b,c,d,e, and f. The restrictions on the arguments a,b,c,d,e and f, and their projections can be seen directly from the defining equation (I-16).

The orthogonality property follows from the unitary transformation

$$\sum_{e} (2e+1)(2f+1)\, W(abcd,ef)\, W(abcd,eg) = \delta_{fg} \qquad \text{(I-18)}$$

A special relation needed elsewhere is

$$W(abcd,0f) = \frac{(-1)^{f-b-d}\, \delta_{ab}\, \delta_{cd}}{[(2b+1)(2d+1)]^{1/2}} \qquad \text{(I-19)}$$

Rotation Matrices

Definition and Properties

Consider a rotation, R, of a system of axis, under which a function $f(x,y.x)$ changes into $f(x', y', z')$. Symbolically, it can be written

$$R\, f(x,y,z) = f(x',y',z') \tag{I-20}$$

As an example, consider a rotation about one axis, for instance, $\hat{z}$ through an angle ϵ. It will be simple to use polar coordinates.

$$R f(r,\theta,\varphi) = f(r,\theta,\varphi-\epsilon) = e^{-\epsilon \frac{\partial}{\partial \varphi}} f(r,\theta,\varphi)\,, \tag{I-21a}$$

or*

$$R f(r,\theta,\varphi) = e^{-i\epsilon L_z} f(r,\theta,\varphi) = e^{-i\epsilon \hat{z}\cdot\vec{L}} f(r,\theta,\varphi). \tag{I-21b}$$

Then for a rotation about an arbitrary axis $\hat{n}$, one can write

$$R(\hat{n},\epsilon) = e^{-i\epsilon \hat{n}\cdot\vec{L}} \tag{I-22a}$$

This operator can be generalized for an arbitrary angular momentum operator $\vec{J}$

$$R(\hat{n},\epsilon) = e^{-i\epsilon \hat{n}\cdot\vec{J}} \tag{I-22b}$$

As is well known, in order to specify a rotation of a system of axis, the Euler angles are often used. These rotations can be described symbolically by the diagram

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* The unit system is chosen such that $\hbar = c = 1$.

$$(x,y,z) \xrightarrow{R(\hat{z},\alpha)} (x_1,y_1,z_1) \xrightarrow{R(\hat{y}_1,\beta)} (x_2,y_2,z_2) \xrightarrow{R(\hat{z}_2,\gamma)} (x',y',z')$$

and the total rotation

$$(x,y,z) \xrightarrow{R(\theta_i)} (x',y',z'), \quad \text{WITH} \quad \theta_i = (\alpha,\beta,\gamma)$$

is given by

$$R(\theta_i) = R(\hat{z}_2,\gamma)\, R(\hat{y}_1,\beta)\, R(\hat{z},\alpha), \tag{I-23a}$$

or

$$R(\theta_i) = e^{-i\gamma J_{z_2}}\, e^{-i\beta J_{y_1}}\, e^{-i\alpha J_z}. \tag{I-23b}$$

Since an unitary transformation U transforms an operator Ω into $U\Omega U^{-1}$, one can write for example

$$e^{-i\beta J_{y_1}} = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\alpha J_z} \tag{I-24}$$

Combining Equation (I-24) and the like with Equation (I-23b), one obtains after simplifications

$$R(\alpha,\beta,\gamma) = e^{-i\alpha J_z}\, e^{-i\beta J_y}\, e^{-i\gamma J_z} \tag{I-25}$$

Now consider an angular momentum representation $|JM\rangle$. Since each representation is uniquely determined by the value of J, a rotation in a three-dimensional space can only connect two different bases with the same value of J, that is:

$$|JM\rangle_{\theta_i} = R(\alpha,\beta,\gamma)|JM\rangle = \sum_{M'} \langle JM'|R(\alpha,\beta,\gamma)|JM\rangle\, |JM'\rangle. \tag{I-26}$$

The definition of the so-called rotation matrix is quite ambiguous, depending on authors. Here the following definition is adopted

$$D^{*J}_{M'M}(\alpha,\beta,\gamma) = \langle JM' | R(\alpha,\beta,\gamma) | JM \rangle , \qquad \text{(I-27a)}$$

or

$$D^{*J}_{M'M}(\alpha,\beta,\gamma) = e^{iM'\alpha} d^{J}_{M'M}(\beta) e^{iM\gamma} , \qquad \text{(I-27b)}$$

where

$$d^{J}_{M'M}(\beta) = \langle JM' | e^{-i\beta J_y} | JM \rangle . \qquad \text{(I-28)}$$

$d^{J}_{M'M}(\beta)$ is real, the explicit expression of which need not be given here, and can be found in standard references[31]. In the special case with $\alpha=0$, $\beta=\pi$ and $\gamma=0$, $D^{J}_{M'M}(\theta_i)$ is given by

$$D^{J}_{M'M}(0,\pi,0) = d^{J}_{M'M}(\pi) = (-1)^{J-M} \delta_{M'\bar{M}} . \qquad \text{(I-29)}$$

As is expected, the matrix D^{*J} is unitary. One has the following equivalent relations

$$\sum_{M} D^{*J}_{MM'} D^{J}_{MM''} = \delta_{M'M''} , \qquad \text{(I-30a)}$$

$$\sum_{M} D^{*J}_{M'M} D^{J}_{M''M} = \delta_{M'M''} . \qquad \text{(I-30b)}$$

Consequently

$$|JM\rangle_{\theta_i} = \sum_{M'} D^{*J}_{M'M} |JM'\rangle , \quad \text{and} \qquad \text{(I-31a)}$$

$$|JM\rangle = \sum_{M'} D^{J}_{MM'} |JM'\rangle_{\theta_i} \quad . \qquad \text{(I-31b)}$$

The coupling rules for the D^J-matrix, known as the Clebsch-Gordon series, are the following

$$D^{J_1}_{M_1M_1'} D^{J_2}_{M_2M_2'} = \sum_{J} \langle J_1 J_2 M_1 M_2 | JM\rangle \langle J_1 J_2 M_1' M_2' | JM'\rangle D^{J}_{MM'} , \qquad \text{(I-32a)}$$

$$D^{J}_{MM'} = \sum_{M_1, M_1'} \langle J_1 J_2 M_1 M_2 | JM\rangle \langle J_1 J_2 M_1' M_2' | JM'\rangle D^{J_1}_{M_1M_1'} D^{J_2}_{M_2M_2'} \quad . \qquad \text{(I-32b)}$$

An important integral is

$$\langle D^{J_3}_{M_3M_3'} | D^{J_2}_{M_2M_2'} | D^{J_1}_{M_1M_1'} \rangle = \frac{8\pi^2}{2J_3+1} \langle J_1 J_2 M_1 M_2 | J_3 M_3\rangle \langle J_1 J_2 M_1' M_2' | J_3 M_3'\rangle , \qquad \text{(I-33)}$$

where the integration is carried over all the Euler angles.

Two special cases are

$$D^{L}_{M'0}(\alpha, \beta, 0) = \left(\frac{4\pi}{2L+1}\right)^{\frac{1}{2}} Y_{LM'}(\beta, \alpha) , \qquad \text{and} \qquad \text{(I-34a)}$$

$$D^{L}_{0M}(0, \beta, \gamma) = (-1)^{M} \left(\frac{4\pi}{2L+1}\right)^{\frac{1}{2}} Y_{L-M}(\beta, \gamma) , \qquad \text{(I-34b)}$$

where Y_{LM} is the spherical harmonic . As a consequence, one obtains the following integral $\langle Y_{L_3M_3} | Y_{L_2M_2} | Y_{L_1M_1} \rangle =$

$$= \left[\frac{(2L_1+1)(2L_2+1)}{4\pi(2L_3+1)}\right]^{\frac{1}{2}} \langle L_1 L_2 00 | L_3 0\rangle \langle L_1 L_2 M_1 M_2 | L_3 M_3\rangle . \qquad \text{(I-35)}$$

Rotation Matrices as Angular Momentum Eigenfunctions

As can be seen in the special cases, Equations (I-34a) and (I-34b), when one of the rotations is absent ($\alpha=0$ or $\gamma=0$) the rotation matrix is the ordinary spherical harmonics. As such, a rotation matrix is also an angular momentum eigenfunction. This property is true even when none of the rotations is absent and will be asserted here without proof:

The rotation matrix $D^J_{MM'}$ is an angular momentum eigenfunction described in a fixed frame and in a rotated frame, respectively.

Let $\vec{J}$ be an infinitesimal generator which generates the rotation $\vartheta_i(\alpha,\beta,\gamma)$, transforming a frame (S) into (S'). Then the following relations hold:

in (S)
$$\begin{cases} J_\pm D^J_{MM'} = \left[(J\mp M)(J\pm M+1)\right]^{\frac{1}{2}} D^J_{M\pm1\,M'}, & \text{(I-36a)} \\ J_3 D^J_{MM'} = M D^J_{MM'}, & \text{(I-37b)} \end{cases}$$

and

in (S')
$$\begin{cases} J'_\pm D^J_{MM'} = \left[(J\pm M')(J\mp M'+1)\right]^{\frac{1}{2}} D^J_{M\,M'\mp1}, & \text{(I-37a)} \\ J'_3 D^J_{MM'} = M' D^J_{MM'}, & \text{(I-37b)} \end{cases}$$

where

$$J_\pm = J_x \pm i J_y \quad ; \quad J'_\pm = J'_x \pm i J'_y$$

Note the difference in signs in Equations (I-36a) and (I-37a). That comes from the fact that in a rotated frame, the angular momentum

operators satisfy

$$[J_i', J_j'] = -i J_k' \quad (i,j,k) \text{ cyclic} \tag{I-38}$$

As is expected, regardless of which frame is chosen for the operator, J^2, one has

$$J^2 D^J_{MM'} = J(J+1) D^J_{MM'} . \tag{I-39}$$

Equations (I-36b), (I-37b) and (I-39) show that the rotation matrix $D^J_{MM'}$ is a simultaneous eigenfunction of J_z, J_z' and J^2 with eigenvalues M, M' and J(J+1), respectively.

Irreducible Tensor Operators

The transformation properties of a tensor are intimately related to the system of coordinates. One then expects to find certain bases in which a set of tensor operators of certain rank transforms in an unique manner. In so doing, one is able to define a set of irreducible tensor operators.

The change of coordinates one often deals with is the rotation in three-dimensional space. The rotation matrix D^J is an irreducible representation of the rotation group O_3 with the infinitesimal generators $J_\pm$ and J_3. If this fact can be used, it will not only be very convenient mathematically, but also provide a better understanding of a few quantum numbers.

Definition

An irreducible tensor operator of rank L is a set of (2L+1) functions $T_{LM}(M=-L,-L+1,\ldots,L)$ which transforms under the rotation, R, according to

$$R T_{LM} R^{-1} = \sum_{M'} D^{*L}_{M'M} T_{LM'} . \qquad \text{(I-40)}$$

For simplicity, the notation T'_{LM} is used instead of $RT_{LM}R^{-1}$. The inverse relation in then given by

$$T_{LM} = \sum_{M'} D^{L}_{MM'} T'_{LM'} . \qquad \text{(I-41)}$$

Another equivalent definition: T_{LM} is an irreducible tensor operator of rank L if it satisfies the commutation relations

$$[J_{\pm}, T_{LM}] = [(L \mp M)(L \pm M + 1)]^{\frac{1}{2}} T_{L\,M\pm1} , \text{ and} \qquad \text{(I-42a)}$$

$$[J_3, T_{LM}] = M T_{LM} . \qquad \text{(I-42b)}$$

The tensor operator defined this way is called the spherical tensor operator.

The addition of two spherical tensors of the same rank is obviously irreducible.

The multiplication and contraction of two spherical tensors $T_{L_1M_1}(\vec{A}_1)$ and $T_{L_2M_2}(\vec{A}_2)$ are defined by

$$T_{LM}(\vec{A}_1,\vec{A}_2) = \sum_{M_1} \langle L_1 L_2 M_1 M_2 | LM \rangle T_{L_1M_1}(\vec{A}_1) T_{L_2M_2}(\vec{A}_2) . \qquad \text{(I-43)}$$

From the definition of the spherical tensor, Equation (I-40), it can be easily shown that $T_{LM}(\vec{A}_1,\vec{A}_2)$ is also irreducible.

The Wigner-Eckart Theorem

The dependence of the matrix element $\langle j'm'|\ T_{LM}\ |jm\rangle$ on the projection quantum numbers is entirely contained in the C-coefficient, that is

$$\langle j'm'|\ T_{LM}\ |jm\rangle = \langle jLmM|j'm'\rangle\langle j'||\ T_L\ ||j\rangle \qquad \text{(I-44)}$$

The quantity $\langle j'\|\ T_L\ \|j\rangle$ is called the reduced matrix element of the operator T_{LM}.

APPENDIX II

In this appendix the expressions for various radial matrix elements, together with some of their numerical values used in the calculations, will be given. However, before writing explicitly $\mathcal{F}_\lambda$ and $\mathcal{F}_\lambda^\pm$, it is instructive to derive the space part of the basic vector $|N\ell\Lambda\Sigma\rangle$ of the intrinsic wave function, X_Ω.

The Normalized Radial Wave Function

Consider the Schrodinger equations

$$\overset{\circ}{H}_0 \Psi = E\Psi \ , \tag{II-1}$$

where $\overset{\circ}{H}_0$ is the Hamiltonian for the three-dimensional isotropic harmonic oscillator defined by

$$\overset{\circ}{H}_0 = \tfrac{1}{2}\omega_0(-\nabla^2+\rho^2) \ . \tag{II-2}$$

Using spherical coordinates, Equation (II-1) can be easily separated. Its angular part yields the well-known spherical harmonics, $Y_{\ell\Lambda}$, and its radial part is

$$R'' + \frac{2}{\rho}R' + \left[\frac{2E}{\omega_0} - \rho^2 - \frac{\ell(\ell+1)}{\rho^2}\right]R = 0 \ . \tag{II-3}$$

With the substitution

$$X = \rho R \ , \quad \text{and} \tag{II-4}$$

$$k^2 = \frac{2E}{\omega_0} \quad , \tag{II-5}$$

the radial equation (II-3) becomes

$$\chi'' + \left[k^2 + \rho^2 - \frac{\ell(\ell+1)}{\rho^2} \right] \chi = 0 \, . \tag{II-6}$$

The solutions of Equation (II-6) are expected to be finite at the origin and at infinity. An inspection of the asymptotic behavior of $X(\rho)$ in Equation (II-6) when $\rho \to 0$ and $\rho \to \infty$ suggests a solution of the form

$$\chi(\rho) = \rho^{\ell+1} e^{-\frac{1}{2}\rho^2} u(\rho) \, . \tag{II-7}$$

With the substitution of Equation (II-7) into Equation (II-6), the differential equation for $X(\rho)$ becomes differential equation for $u(\rho)$, and reads

$$u'' + 2\left(\frac{\ell+1}{\rho} - \rho\right) u' - \left[2(\ell + \tfrac{3}{2}) - k^2 \right] u = 0 \, . \tag{II-8}$$

With the change of variable

$$x = \rho^2 \tag{II-9}$$

Equation (II-8) transforms into a confluent differential equation

$$x \frac{d^2u}{dx^2} + \left[(\ell + \tfrac{3}{2}) - x \right] \frac{du}{dx} + \left[\tfrac{1}{2}(\ell + \tfrac{3}{2}) - \frac{k^2}{4} \right] u = 0 \, . \tag{II-10}$$

A solution of Equation (II-10) is the confluent hypergeometric function[32]

$$u(x) = {}_1F_1\left[\tfrac{1}{2}(\ell+\tfrac{3}{2}) - \tfrac{k^2}{4}\,,\ \ell+\tfrac{3}{2},\ x\right]\,. \qquad \text{(II-11)}$$

The condition that $R(\rho)$ is decreasing at infinity requires $1/2(\ell+3/2)-\frac{k^2}{4}$ be a negative integer or zero. Hence

$$\tfrac{1}{2}(\ell+\tfrac{3}{2}) - \tfrac{k^2}{4} = -n \quad ,(n = 0,1,2,\ldots)\,. \qquad \text{(II-12)}$$

Combining Equations (II-5) and (II-12), one obtains for the energy eigenvalues

$$E_{N\ell} = (N+\tfrac{3}{2})\omega_0\,, \text{ and} \qquad \text{(II-13a)}$$

$$N = 2n+\ell\,. \qquad \text{(II-13b)}$$

Equations (II-4), (II-7), (II-9) and (II-11) give the corresponding eigenfunction

$$R_{n\ell}(\rho) = C_{n\ell}\,\rho^{\ell} e^{-\frac{1}{2}\rho^2}\,{}_1F_1(-n,\ell+\tfrac{3}{2},\rho^2)\,, \qquad \text{(II-14)}$$

where $C_{n\ell}$ is a constant of integration.

Now, the eigenfunction $R_{n\ell}(\rho)$ is normalized to unity, i.e.,

$$|C_{n\ell}|^2\int_0^\infty \rho^{2\ell} e^{-\rho^2}\left[{}_1F_1(-n,\ell+\tfrac{3}{2},\rho^2)\right]^2\rho^2\,d\rho = 1\,. \qquad \text{(II-15)}$$

With the change of variable (II-9), the integral (II-15) becomes

$$\tfrac{1}{2}|C_{n\ell}|^2\int_0^\infty x^{\ell+\frac{1}{2}} e^{-x}\left[{}_1F_1(-n,\ell+\tfrac{3}{2},x)\right]^2 dx = 1\,. \qquad \text{(II-16)}$$

The hypergeometric function ${}_1F_1(-n,\ell+3/2,x)$ can be written, in terms of the Laguerre polynomials, $L_n^{\ell+1/2}(x)$,

$$ {}_1F_1(-n,\ell+\tfrac{3}{2},x) = \frac{n!\,\Gamma(\ell+\frac{3}{2})}{\Gamma(n+\ell+\frac{3}{2})}\, L_n^{\ell+\frac{1}{2}}(x), \qquad \text{(II-17)} $$

and since[32]

$$ \int_0^\infty x^{\ell+\frac{1}{2}} e^{-x} \left[L_n^{\ell+\frac{1}{2}}(x) \right]^2 dx = \frac{\Gamma(n+\ell+\frac{3}{2})}{n!}, \qquad \text{(II-18)} $$

one deduces for the normalizing factor $C_{n\ell}$ from Equations (II-16), (II-17) and (II-18)

$$ C_{n\ell} = \left[\frac{2\,\Gamma(n+\ell+\frac{3}{2})}{n!} \right]^{1/2} \frac{1}{\Gamma(\ell+\frac{3}{2})} . \qquad \text{(II-19)} $$

Equation (II-14), with the coefficient $C_{n\ell}$ given by Equation (II-19), is the normalized radial wave function $|N\ell\rangle$ in $|N\ell\wedge\Sigma\rangle$.

Note that the variable r used in various operators (Chapter IV) is not the same as ρ here, but related by

$$ r = (M\omega_0)^{-\frac{1}{2}} \rho . \qquad \text{(II-20)} $$

Now the radial integrals $\mathcal{T}_\lambda$ and $\mathcal{T}_\lambda^\pm$ can be evaluated. To be more precise, they will be written as $\mathcal{T}_\lambda(N_i\ell_i, N_f\ell_f)$ and $\mathcal{T}_\lambda^\pm(N_i\ell_i, N_f\ell_f)$.

Expression for $\mathcal{T}_\lambda(N_i \ell_i , N_f \ell_f)$

The radial matrix element $\langle N_f \ell_f | \rho^\lambda | N_i \ell_i \rangle$ can be evaluated by deduction. However, only the result is given here.

$$\langle N_f \ell_f | \rho^\lambda | N_i \ell_i \rangle = \frac{\Gamma(n_i+1)\Gamma(n_f+1)}{\Gamma(n_i+t-\nu+1)\Gamma(n_f+t-\nu'+1)} \nu!\,\nu'! \times \sum_\sigma \frac{\Gamma(t+\sigma+1)}{\sigma!\,(n_i-\sigma)!\,(n_f-\sigma)!\,(\sigma+\nu-n_i)!\,(\sigma+\nu'-n_f)!} , \qquad \text{(II-21)}$$

$$n_i = \tfrac{1}{2}(N_i-\ell_i) \quad , \quad n_f = \tfrac{1}{2}(N_f-\ell_f) , \qquad \text{(II-22a)}$$

$$\nu = \tfrac{1}{2}(\ell_f-\ell_i+\lambda) \quad , \quad \nu' = \tfrac{1}{2}(\ell_i-\ell_f+\lambda) , \qquad \text{(II-22b)}$$

$$t = \tfrac{1}{2}(\ell_i+\ell_f+\lambda+1) \qquad \text{(II-22c)}$$

$$\begin{matrix} n_i \\ n_f \end{matrix} \geqslant \sigma \geqslant \begin{matrix} n_i-\nu \\ n_f-\nu' \end{matrix} . \qquad \text{(II-22d)}$$

The integral (II-21) vanishes unless σ satisfies Inequality (II-22d). The equivalent conditions for the matrix element of ρ^λ to be different from zero are

$$\ell_i+\lambda \geqslant \ell_f \geqslant \ell_i-\lambda , \qquad \text{(II-23a)}$$

or

$$N_i+\lambda \geqslant N_f \geqslant N_i-\lambda . \qquad \text{(II-23b)}$$

Combining Equations (II-20) and (II-21), one obtains for $\mathcal{T}_\lambda(N_i \ell_i , N_f \ell_f)$

$$\mathcal{T}_\lambda(N_i \ell_i , N_f \ell_f) = (M\omega_0)^{-\lambda/2} \langle N_f \ell_f | \rho^\lambda | N_i \ell_i \rangle \qquad \text{(II-24)}$$

A few numerical values of Equation (II-21), needed in the computation, are given in Table 14 at the end of this appendix.

Expressions for $\mathcal{J}_\lambda^{\pm}(N_i \ell_i, N_f \ell_f)$

Now consider the radial matrix elements $\langle N_f \ell_f | \rho^\lambda D_\pm(\ell_i) | N_i \ell_i \rangle$ where $D_\pm(l)$ is defined by

$$D_+(\ell) = \frac{d}{d\rho} + \frac{\ell+1}{\rho} \quad \text{and} \quad D_-(\ell) = \frac{d}{d\rho} - \frac{\ell}{\rho} . \tag{II-25}$$

Letting $\frac{d}{d\rho}$ operate on $|N\ell\rangle$, and making use of Equation (II-19) and of the relation

$$\frac{d}{dx} {}_1F_1(\alpha, \beta, x) = \frac{\alpha}{\beta} {}_1F_1(\alpha+1, \beta+1, x) , \tag{II-26}$$

one obtains

$$\frac{d}{d\rho} |N\ell\rangle = \ell \rho^{-1} |N\ell\rangle - \rho |N\ell\rangle - 2\sqrt{n} |N-1\ \ell+1\rangle . \tag{II-27}$$

Equations (II-25) and (II-27) then give

$$D_+ |N\ell\rangle = (2\ell+1) \rho^{-1} |N\ell\rangle - \rho |N\ell\rangle - 2\sqrt{n} |N-1\ \ell+1\rangle , \tag{II-28a}$$

and

$$D_- |N\ell\rangle = -\rho |N\ell\rangle - 2\sqrt{n} |N-1\ \ell+1\rangle . \tag{II-28b}$$

Combining Equations (II-21) and (II-28), the radial matrix elements $\langle N_f \ell_f | \rho^\lambda D_\pm | N_i \ell_i \rangle$ will be given by

$$\langle N_f \ell_f | \rho^\lambda D_+ | N_i \ell_i \rangle = (2\ell_i + 1) F_{\lambda-1}(N_i \ell_i, N_f \ell_f) - F_{\lambda+1}(N_i \ell_i, N_f \ell_f)$$

$$- \left[2(N_i - \ell_i)\right]^{1/2} F_\lambda(N_i - 1\ \ell_i + 1, N_f \ell_f) , \quad \text{and} \tag{II-29a}$$

$$\langle N_f \ell_f | \rho^\lambda D_- | N_i \ell_i \rangle = - F_{\lambda+1}(N_i \ell_i, N_f \ell_f) - \left[2(N_i - \ell_i)\right]^{1/2} F_\lambda(N_i - 1\ \ell_i + 1, N_f \ell_f), \quad \text{(II-29b)}$$

where

$$F_\lambda(N_i \ell_i, N_f \ell_f) = \langle N_f \ell_f | \rho^\lambda | N_i \ell_i \rangle \ . \quad \text{(II-29c)}$$

Combining Equations (II-20) and (II-2a), one obtains for $\mathcal{F}_\lambda^{\pm}(N_i \ell_i, N_f \ell_f)$

$$\mathcal{F}_\lambda^{\pm}(N_i \ell_i, N_f \ell_f) = (M\omega_0)^{\frac{1-\lambda}{2}} \langle N_f \ell_f | \rho^\lambda D_\pm | N_i \ell_i \rangle \ . \quad \text{(II-30)}$$

Some numerical values of Equations (II-29) with $\lambda=0$ are given in Table 14.

Table 14. Numerical Values of the Radial Integrals

N_i	l_i	$\langle N_f \ell_f \vert \rho \vert N_i \ell_i \rangle$	$\langle N_f \ell_f \vert D_{\pm}(\ _i) \vert N_i \ell_i \rangle$
$N_f + 1$	$\ell_f + 1$	$[\frac{1}{2}(N_f+\ell_f+3)]^{1/2}$	$-[\frac{1}{2}(N_f+1_f+3)]^{1/2}$
$N_f + 1$	$\ell_f - 1$	$[\frac{1}{2}(N_f-\ell_f+2)]^{1/2}$	$-3[\frac{1}{2}(N_f+1_f+2)]^{1/2}$
$N_f - 1$	$\ell_f + 1$	$[\frac{1}{2}(N_f-\ell_f)]^{1/2}$	$-[\frac{1}{2}(N_f-1_f)]^{1/2}$
$N_f - 1$	$\ell_f - 1$	$[\frac{1}{2}(N_f+\ell_f+1)]^{1/2}$	$-[\frac{1}{2}(N_f+\ell_f+1)]^{1/2}$

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